ABSTRACTS

Local-to-global Frames and Applications to Dynamical Sampling Problem

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We consider systems of vectors in a Hilbert space $\mathcal{H}$ of the form $\{g_{jk} : j \in J, k \in K\} \subset \mathcal{H}$ where J and K are countable sets of indices. We find conditions under which the local reconstruction properties of such a system extend to global stable recovery properties on the whole space. As a particular case we obtain new local-to-global results for systems of type $\{A^n g\}_{g\in\mathcal{G},0\leq n\leq L}$ arising in the dynamical sampling problem.

Optimal Recovery of Monotone Operators in Quasinormed Spaces

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The concept of quasinormed space was introduced by S. Aseev in 1985. The axioms of a quasinormed space are satisfied for example by the space of closed bounded subsets of an arbitrary Banach space, as well as many spaces of fuzzy sets. We consider the problem of optimal recovery of monotone operators in the space of functions with values in quasinormed spaces. In particular, we obtain a wide generalization of Kiefer's well-known result on the optimal recovery of integrals of monotone functions defined on a finite interval.

On Multivariate Ostrowski Type Inequalities and Their Applications to Optimal Recovery of Integrals

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In 1938 Ostrowski proved the following result. Let $f\colon [-1,1] \to \mathbb{R}$ be a differentiable function and let $|f'(t)| \leq 1$ for all $t \in (-1,1)$. Then for all $x \in [-1,1]$ the inequality $|\frac{1}{2}\int_{-1}^{1} f(t)dt - f(x)| \leq (1+x^2)/2$ holds. The inequality is sharp in the sense that for each fixed $x \in [-1,1]$ the upper bound $\frac{1+x^2}{2}$ cannot be reduced. This result gave rise to a special branch in the Theory of Inequalities, namely inequalities that estimate the deviation of the value of a function from its mean value with the help of some characteristics of the function. Such inequalities are now called *Ostrowski type inequalities*. In this talk we present sharp Ostorowski type inequality for multivariate Sobolev classes and apply it to the problem of optimal recovery of integrals.

A Spline Theory of Deep Networks

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We build a rigorous bridge between deep networks (DNs) and approximation theory via spline functions and operators. Our key result is that DNs based on piecewise-affine, convex nonlinearities can be written as a composition of max-affine spline operators (MASOs), which provide a powerful portal through which to view and analyze their inner workings.

The spline partition of the input signal space that is implicitly induced by a MASO directly links DNs to the theory of vector quantization (VQ), Voronoi/power diagrams, and K-means clustering, which opens up new geometric avenue to study how DNs organize signals in a hierarchical fashion. For instance, the output of a MASO DN can be written as a simple VQ-region-dependent affine transformation of the input. This implies that a DN constructs a set of signal-dependent, class-specific templates against which the signal is compared via a simple inner product; we explore the links to the classical theory of optimal classification via matched filters and the effects of data memorization. In particular, we provide a formula for how the composition of layers (e.g., the entire DN) produces a progressively subdivided power diagram of the signal input space. The MASO framework extends to DNs using a large class of smooth nonlinearities beyond piecewise-affine, convex ones via a simple probabilistic argument. To illustrate, using a Gaussian mixture model, we show that ReLU, absolute value, and max-pooling can be interpreted as solutions to certain natural "hard" (deterministic) VQ inference problems, while sigmoid, hyperbolic tangent, and softmax can be interpreted as solutions to corresponding "soft" (probabilistic) VQ inference problems.

On RBF-FD Approximations Augmented with High-degree Polynomials

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Radial basis function-generated finite differences (RBF-FD) based on the combination of polyharmonic splines (PHS) with high degree polynomials (PHS+poly) have recently emerged as a powerful and robust numerical approach for the local interpolation and derivative approximation of functions over scattered node layouts. One key feature is that high orders of accuracy can be achieved without the need of selecting a shape parameter or the issues related to numerical ill-conditioning. Specially striking is the behavior of the interpolant near domain boundaries, where regular FD approximations of high accuracy will have very large weights well into the domain due to Runge's phenomenon. The inclusion of PHS-type RBFs in the process of generating RBF-FD weights makes it possible to avoid this adverse effect. This talk provides an insight into these features, focusing on heuristic perspectives and numerical demonstrations.

Approximation of Bivariate Functions by Filtered Back Projection

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We consider the approximation of bivariate functions by the method of filtered back projection, which is a commonly used reconstruction technique in computerized tomography and allows to recover an unknown target function from its given Radon samples. The reconstruction is based on the classical filtered back projection (FBP) formula, which gives an analytical inversion of the Radon transform. The FBP formula, however, is numerically unstable and suitable low-pass filters of finite bandwidth and with a compactly supported window function are typically employed to make the reconstruction by FBP less sensitive to noise. This leads to an *approximate* reconstruction of the target function.

In this talk we analyse the inherent FBP approximation error which is incurred by the application of the low-pass filter. To this end, we present error estimates in Sobolev spaces of positive fractional order, where the obtained error bounds depend on the bandwidth of the utilized filter, on the flatness of the filter's window function at the origin, on the smoothness of the target function, and on the order of the considered Sobolev norm. Furthermore, we prove convergence for the approximate FBP reconstruction in the treated Sobolev norms along with asymptotic rates of convergence as the filter's bandwidth goes to infinity, where we observe saturation at fractional order depending on smoothness properties of the filter's window function. The theoretical results are supported by numerical experiments.

Metric Approximation of Set-valued Functions of Bounded Variation

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We study approximation of set-valued functions (SVFs) — functions mapping a real interval to compact sets in $\mathbb{R}^d$. In addition to the theoretical interest in this subject, it is relevant to various applications in fields where SVFs are used, such as economy, optimization, dynamical systems, geometric modeling. The earlier works in this area are mainly concerned with approximation of set-valued functions with convex images, for which the tools of Minkowski linear combinations and the Aumann integral are effective. Yet these techniques posses the property of convexification: the resulting approximation is always a function with convex images, even if the function to be approximated is not. Clearly, such methods are useless for the approximation of SFVs with general, not necessarily convex images. Dyn, Farkhi and Mokhov developed in a series of work a new approach that is free of convexification — the so-called metric linear combinations and the metric integral. Adaptations of classical approximation operators to continuous SFVs were studied by Dyn, Farkhi and Mokhov. Here, we develop methods for approximation of SFVs that are not necessarily continuous. We consider SVFs of bounded variation in the Hausdorff metric. In particular, we adapt to SVFs local operators such as the symmetric Schoenberg spline operator, the Bernstein polynomial operator and the Steklov function. Error bounds, obtained in the averaged Hausdorff metric, provide rates of approximation similar to those for real-valued functions of bounded variation.

Adaptive Approximations on Conforming Partitions

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We consider piecewise approximations on general triangulations motivated by the finite element methods (FEM) for numerically solving partial differential equations. Finding an approximation with the smallest error for a given number of degrees of freedom is usually computationally prohibitive and we set for finding a near-best approximation instead. Results for such approximations are usually formulated as tree approximations relating the process of refining the partition to a decision tree and emphasizing the fact that the algorithms are coarse-to-fine. These results, however, feature general nonconforming partitions allowing hanging nodes, i.e. cases in which the common boundary of two neighboring triangles is not a side of both of them. The usual adaptive strategy for finding conforming partitions in FEM is "mark $\rightarrow$ subdivide $\rightarrow$ complete". In this strategy any element can be marked for subdivision but since the resulting partition often contains hanging nodes, additional elements have to be subdivided in the completion step to get a conforming partition. This process is very well understood for triangulations received via newest vertex bisection procedure. In particular, it is proven that the number of elements in the final partition is limited by constant times the number of marked cells.

This motivated us to design a marking procedure that is limited only to cells of the partition whose subdivision will result in a conforming partition and therefore no completion step is necessary. We also proved that this procedure is near-best in terms of both error of approximation and complexity with efficient constants.

Optimal Multivariate Spline Method for Recovery of Functions of Smoothness Three

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We construct an optimal multivariate spline method for global recovery of functions from the class $W^3(P)$ of functions that are C^2-continuous in a neighborhood of a given d-dimensional convex polytope P and have uniformly bounded third-order derivatives in any direction on P. This method uses as information data of smoothness two: the values of functions and their first and second order partial derivatives at a fixed finite point set $X \subset P$, which includes the vertices of P. We show the optimality of this method among all non-adaptive recovery algorithms. The recovery error is measured in the uniform norm on P. The optimal method turns out to be a continuous spline of degree three over a Delaunay triangulation of the set X in P, which interpolates function values at X. The solution for the periodic analogue of this optimal recovery problem is also obtained.

An optimal recovery method based on the data of smoothness one was earlier constructed by the author and T.S. Sorokina (2011) for a similar class $W^2(P)$ of functions of smoothness two on P and by the author for the above mentioned class $W^3(P)$ (2018).

BPX Preconditioners for Isogeometric Analysis based on Hierarchical Splines

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Hierarchical splines are among the most widely used types of splines breaking the tensor-product structure of B-splines, and therefore allowing local refinement. For this reason, they have been extensively used to construct adaptive schemes for the solution of partial differential equations in the context of isogeometric analysis. In this talk, we focus on the construction of BPX preconditioners for the solution of the linear system needed in the adaptive isogeometric schemes with hierarchical splines. The preconditioners are based on decompositions of the hierarchical space. We proved that the condition number is bounded with respect to the number of levels, under the condition that the mesh is admissible, which is equivalent to a suitable grading. This is achieved, similarly to adaptive schemes based on T-splines, by using stable decompositions satisfying the Strengthened Cauchy-Schwarz inequality. Several examples highlighting the role of the decomposition, of the basis of the hierarchical space and of the admissibility of the mesh will be shown.

Improving Bayesian Quadrature for Constrained Integrands

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For many applications of Bayesian quadrature, practitioners have *a priori* knowledge about certain aspects of the integrand, such as its range. This talk presents an improved Bayesian framework for performing inference of affine transformations of such constrained functions, with a focus on quadrature of nonnegative functions, a common task in Bayesian inference. We present explicit approximation schemes for both nonnegativity and boundedness constraints, and argue for the use of a log transformation for functions with high dynamic range such as likelihood surfaces. We propose a novel method for optimizing hyperparameters associated with this framework where the marginal likelihood is maximized in the original space, as opposed to in the transformed space. The result is a model that better explains the actual data. Experiments on synthetic and real-world data demonstrate our framework achieves superior estimates using less wall-clock time than existing Bayesian quadrature procedures. In addition to quadrature tasks, we showcase the flexibility of our framework by applying it to a model selection task, where it also achieves more accurate estimates of posterior model probabilities than standard Bayesian quadrature.

An Algorithm for Shape Preserving Approximation

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We present an algorithm of joint approximation of a function and its first derivative by alternative orthogonal polynomials on the interval $[0, 1]$ and construct an operator for shape preserving approximation of the function. The operator has n-th degree polynomial reproduction property and exhibits spectral convergence for smooth functions. Exponential and rational reformulations of the algorithm can result in shape preserving approximation on a half-line. The algorithm may be of interest for isogeometric analysis and spectral methods. Numerical examples on approximation will be given.

Phase Retrieval of Complex and Vector-valued Signals

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Phase retrieval problem considers whether and how one can recover the signal, up to a trivial ambiguity, from its magnitude measurements. In this talk, we first characterize the conjugate phase retrieval of signals in a complex linear space. We then consider the phase retrieval for a real vector-valued signal, that is determining the signal, up to an orthogonal matrix, from its magnitude measurements. We introduce a notion to the nonseparable signal and establish the equivalence between the phase retrievability of a vector-valued signal and its nonseparability. We also analyze and compare the conjugate phase retrieval for complex signals and phase retrieval for real vector-valued signals.

Manifold Learning with Diffusion Variational Autoencoders

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Variational autoencoders (VAEs) have become one of the most popular deep learning approaches to unsupervised learning and data generation. However, traditional VAEs suffer from the constraint that the latent space distributionally match an iid normal distribution, independent of the initial data

distribution. This leads to a number of issues around modeling manifold data, as there is no function with bounded Jacobian that maps a normal distribution to certain manifolds (e.g. sphere). Similarly, there are not many theoretical guarantees on the encoder and decoder created by the VAE. In this work, we propose a variational autoencoder that maps manifold valued data to its diffusion map coordinates in the latent space, resamples in a ball around a given point in the latent space, and learns a decoder that maps the newly resampled points back to the manifold. The framework is built off of SpectralNet, and is capable of learning this data dependent latent space without computing the eigenfunction of the Laplacian explicitly. We prove that the diffusion variational autoencoder framework is capable of learning a locally bi-Lipschitz map between the manifold and the latent space, and that our resampling method around a point in the latent space $\phi(x)$ maps points back to the manifold around the point x, specifically into a ball on the tangent space at the point x on the manifold. We also provide empirical evidence of the benefits of using a diffusion map latent space, and even show benefits of the framework on non-manifold data.

Non-stationary Subdivision Schemes: Potentialities and Perspectives

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Subdivision schemes were created originally to design geometrical models but very soon they were recognized as methods for approximation. They are iterative methods for points generation based on refinement rules that can be easily and efficiently implemented on a computer.

Since the 90s, subdivision schemes attracted many scientists for both the simplicity of their basic ideas and the mathematical elegance emerging in their analysis: they are defined by repeatedly applying simple and local refinement rules which have been extended to refine other object such as vectors, matrices, sets of points, curves, nets of functions or manifold data. Therefore, the domain of application of subdivision is vast and they emerge in different contexts ranging from computer animation to motion analysis.

The most studied subdivision schemes are *linear* and *stationary* (level independent). A nice aspect of linear subdivision schemes is that many of their properties can be translated into algebraic properties of Laurent polynomials. This make the verification of some of their properties easy. Moreover, since these schemes can be viewed as repeated multiplication by matrices, many analysis tools are based on linear algebra such as the "joint spectral radius of two matrices. Stationary schemes are characterised by repeatedly applying the same simple and local refinement rule while the non-stationary (level dependent) schemes apply a different rule in each level of refinement. But, the non-stationary schemes are equally simple to implement, and highly intuitive in use: from an implementation point of view changing rules with the levels is not a crucial matter, also in consideration of the fact that, in practice, only few subdivision iterations are performed.

Non-stationary subdivision schemes were originally introduced with the aim of enriching the class of limit functions of stationary schemes. For example, in the univariate case, they can generate exponential B-splines or C^∞compactly supported functions like the Rvachev function. Actually these schemes have very different and distinguished properties: while stationary subdivision schemes are not capable of generating conic sections, or to deal with level-dependent tension parameters to modify the shape of a subdivision limit, non-stationary schemes are able to. Moreover, level-dependent subdivision schemes include Hermite schemes that do not model only curves and surfaces, but also their gradient fields, and are considered of interest both in geometric modelling and biological imaging. Additionally, non-stationary subdivision schemes are at the base of the construction of non-stationary wavelet and framelets that, being level adapted, are certainly more flexible. Last, but not least, level-dependent rules could be used to overcome the standard limitations of subdivision surfaces such as artefacts and low regularity at extraordinary vertices/faces.

Pointwise and Uniform Convergence of Fourier Extensions

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A Fourier extension approximation to a function on $[-1, 1]$ is a trigonometric polynomial whose period is $T > 1$, which removes the periodicity constraint in $[-1, 1]$. In this talk we illustrate earlier L^2 norm error convergence results and discuss recent pointwise and uniform convergence results when the approximation is taken to minimise the $L^2(-1, 1)$ error. These show that the pointwise convergence of Fourier extensions is more similar to Legendre series than classical Fourier series. In particular, unlike classical Fourier series, Fourier extensions yield pointwise convergence at the endpoints of the interval. The proof connects Fourier extensions to Legendre polynomials which are orthogonal on an arc of the complex unit circle, and uses Lebesgue's Lemma for to bound pointwise error this standard technique in constructive approximation theory involves bounding the Lebesgue function and best uniform approximation estimates. From a computational perspective, Fourier extensions are too ill-conditioned to compute because there is an inherent redundancy. A regularisation is used in practice, leading us to pose some open questions regarding convergence in this practically applicable case.

Interpolation of PDE Data Using a Designer Kernel

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The Laguerre polynomials were previously used to define a nonstationary, time-dependent kernel via its Mercer series. We extend this idea to create a parameterized family of time-dependent kernels with variable smoothness, similar to the Matérn family. The nonstationarity of the kernels allow for modeling where system behavior is dictated by an initial condition, possibly unknown to the user. We show the construction of this family of kernels, and demonstrate it on data generated from parabolic partial differential equations. We can encode the spatial structure of the data by combining with kernels which respect boundary conditions. The construction of the Laguerre family and the iterated Brownian bridge kernel are contrasted, but for both we have immediate access to the eigenfunctions and eigenvalues. Thus we also demonstrate the use of the Hilbert Schmidt SVD (HS-SVD) for computations in the flat limit, and discuss some implementation details of the HS-SVD in the case of a tensor product of kernels.

Error Bounds for a Least Squares Meshless Finite Difference Method on Closed Manifolds

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Meshless finite difference methods discretize a differential equation $Lu = f$ on a finite set $X = \{x_1, \dots, x_N\} \subset \Omega$ with the help of local numerical differentiation formulas, and seek a discrete solution $\hat{u} \in \mathbb{R}^N$ such that $\hat{u} \approx u|_X$, obtained by solving sparse linear systems of equations. The presentation will be devoted to the convergence analysis of a least squares version of this method, giving for the first time sufficient conditions on a set of discretization nodes X and sets of influence that guarantee convergence $\|\hat{u} - u|_X\| \to 0$ as $N \to \infty$ in meshless setting. The results apply to the case when $\Omega = \mathcal{M}$ is a smooth closed manifold, numerical differentiation folmulas are obtained with the help of a reproducing kernel for a Sobolev space on $\mathcal{M}$, L belongs to a certain class $\mathcal{L}$ of elliptic differential operators, and

u is sufficiently smooth. The class $\mathcal{L}$ includes in particular $L = -\Delta_{\mathcal{M}} + \omega I$, $\omega > 0$, where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on the d-dimensional sphere $\mathcal{M} = \mathcal{S}_d$ or another homogeneous manifold.

A Deep Neural Network Architecture Inspired by Polynomial Approximation

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We introduce a class of deep neural networks whose architecture is inspired by polynomial approximation and which can be used for function approximation. Deep neural networks have been deployed for solving many challenging tasks. Despite the immense amount of recent research on neural networks choosing a network architecture and initialization for a given problem is difficult. We show that the parameters of our proposed network can be initialized so that it approximates a given polynomial function. If the chosen polynomial is itself an approximation of a function of interest, our network performs at least as well as the polynomial approximation. We then consider training our constructed network and show several examples where the network achieves an approximation better than the polynomial approximation used to initialize its parameters.

The Natural Greedy Algorithm for Reduced Bases in Banach Spaces

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The goal of a reduced basis method is to find an approximating subspace for a given set of data. We present the Natural Greedy Algorithm — a novel way of constructing reduced bases in Banach spaces that utilizes the norming functionals of the basis elements in order to project onto subspaces. Such an approach allows for a significantly simpler basis construction as compared to the classical Greedy Algorithm, which commonly computes projections by solving a high-dimensional minimization problem. As it turns out, the performance of the Natural Greedy Algorithm is similar to that of the Greedy Algorithm in terms of both theoretical and numerical results, while the realization of the latter is substantially more computationally expensive in general. In addition, we compare our algorithm to the other two popular reduced bases techniques: the Proper Orthogonal Decomposition and the Empirical Interpolation Method.

High-dimensional Function Approximation with ReLU Deep Neural Networks

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Over the past decade, advances in software architectures and specialized hardware have enabled *machine learning* (ML) with *deep neural networks* (DNNs) to achieve impressive results on many historically challenging problems, e.g., image classification, autonomous vehicles, and speech recognition. Driving these successes is the modern ubiquity of large-scale data sets for training and feature detection, and methods for their processing and meta-data capture, a data-science paradigm referred to as *big data*. DNNs are also increasingly being applied to problems in the sciences, such as approximating

solutions to *partial differential equations* (PDEs) defined on high-dimensional domains. Recent theoretical results on DNNs show that such architectures allow for the same approximation rates as best in class schemes such as "hp-adaptive" finite element and spectral approximations in norms relevant to PDE problems.

However, despite these results, many open questions related to obtaining such approximations in practice remain. In this talk we present results on the application of a standard DNN architecture, namely the fully-connected feedforward neural network with ReLU activation function $\sigma(x) = \max\{x, 0\}$, to the problem of finding an approximation to a function f, defined on a compact domain $\mathcal{U} \subset \mathbb{R}^d$, from a set $(x_i, f(x_i))_{i=1}^m$ of points and function values. We also present results on the existence of a DNN and training procedure which performs as well as *compressed sensing* (CS) for high-dimensional function approximation in terms of accuracy and sample complexity. Hence, to provide a comparison, we include results obtained with CS on the same problems. In particular, our results highlight key issues of stability in the training process and the amount of data required to obtain an approximation.

Improving Dimension Bounds for Splines: The Homological Term

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We introduce a combinatorial improvement to Schumaker's lower bound for splines on planar rectilinear partitions which also applies to trivariate splines on cells. This improvement applies in many situations where there are interior vertices (resp. edges) that are surrounded by interior edges (resp. two-faces) that determine coincident slopes; when this happens we can improve existing bounds using a combinatorial contribution from a homology module studied by Billera, Schenck, and Stillman. Combined with upper bounds due to Schumaker, Mourrain, and Villamizar, this sometimes allows the determination of exact dimension formulas in low degree. We give several illustrations of this improvement, including confirming a recent conjecture for the dimension of C^1 splines on bipyramid cells by Colvin, DiMatteo, and Sorokina.

Lipschitz Analysis of Noisy Quantum Inference as Phase Retrieval

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Many physical systems are best described as statistical ensembles over pure quantum states, the standard example being un-polarized light. In this case the problem of inference is to determine the density matrix of the system via repeated measurements of a well chosen collection of observables. We show that for finite dimensional quantum states this problem is equivalent to a particular generalization of the well studied phase retrieval problem. We analyze the geometry of the underlying mathematical spaces and provide a stability analysis of the phase retrieval problem in this case. Moreover, we show that the number of observables must be greater than $2nr - r^2$ in order to infer the density matrix of an ensemble of r quantum states of dimension n.

Approximation of Iterative Function Systems Through Neural Network Functions

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As part of the endeavor to understand the underlying mechanisms of neural networks in recent years, several works studied the number of linear regions ℓ of piecewise linear functions generated by deep neural networks (DNN). In particular, it has been shown that ℓ can grow exponentially with the number of network parameters p. This property is often used as a possible explanation of the advantages of DNNs over shallow networks in approximating complicated functions. Nonetheless, a simple dimension argument shows that DNNs cannot span all piecewise linear functions with ℓ linear regions, for $\ell > p$. Thus, it is natural to strive to characterize specific families of functions with ℓ linear regions which can be constructed by DNNs.

Fractals originating from Iterative Function Systems (IFS) can be realized through a sequence of piecewise linear functions F_k with a number of linear regions exponential in k. We show that, under some mild assumptions, F_k can be represented by a neural network function comprising of $\mathcal{O}(k^2)$ parameters only. IFS have been successfully used for image compression. Albeit its commercial failure, the fractal-based compression shows that the human visual system is locking on self-similarities in images, to some extent. This fact, combined with the approximation scheme of IFS through neural network functions presented here, may give some intuition to the reason of the undoubted success of DNNs in image processing tasks.

Recovering Partially Known Signals from Few Fourier Intensity Measurements

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It is well known that a generic signal can be recovered from its fourier intensity function and knowledge of a single entry. Suppose that we only know a few fourier intensity measurements but know more entries of the signal. Can we recover the signal from this data? We prove that if the known entries are concentrated then it is possible recover a signal from relatively few fourier intensity measurements. These results were applied by the authors to show that generic signals can be recovered from relatively few phaseless, blind STFT measurements?

Geometry, Mesh Parameterization, and High-order Spline Approximation

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High-order spline approximations harbor the potential to deliver improved accuracy per degree-of-freedom versus low-order approximations. However, when applied to curved geometries, their success hinges upon the use of a curvilinear mesh of not only sufficiently high accuracy but also sufficiently high quality. While the impact of mesh size, polynomial degree, and continuity on the resolving power of a high-order spline approximation has been extensively studied in the past, the impact of mesh parameterization has been relatively understudied.

In this talk, theoretical results are presented quantifying the impact of mesh parameterization on the accuracy of a high-order spline approximation, and a formal definition of shape regularity is introduced for curvilinear meshes based on these results. This formal definition of shape regularity in turn inspires a new set of quality metrics for curvilinear meshes. Computable bounds are established for these quality metrics using the Bernstein-Bézier form, and a new curvilinear mesh optimization procedure is proposed based on these bounds. Numerical results confirming the importance of shape regularity in the context

of high-order finite element methods are presented, and numerical results demonstrating the promise of the proposed curvilinear mesh optimization procedure are also provided. The theoretical results presented in this talk apply to any piecewise-polynomial or piecewise-rational approximation posed on a mesh of polynomial or rational mapped simplices and hypercubes. As such, they apply not only to classical continuous Galerkin finite element methods but also to discontinuous Galerkin finite element methods and even isogeometric methods based on NURBS, T-splines, or hierarchical B-splines. This is joint work with L. Engvall.

Reconstructing Analog Signals from Discrete Measures: A General Regularization-based Approach

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We consider the recovery of an input analog signal from finitely many and possibly noisy discrete measurements. The reconstruction task is formulated as an optimization problem. The data-fidelity constraint conducts to a dramatically ill-posed problem due to the infinite-dimensional nature of the underyling signal. We introduce a general regularization-based framework to address the ill-posedness issue. The regularization allows for imposing structure priors on the signal to reconstruct, besides making the problem well-posed. A crucial mathematical aspect is to specify the domain of definition of the problem, in order to have the most general formulation, what will be the subject of this talk. We will cover traditional Hilbertian quadratic regularizations (Tikhonov, RKHS) together with recently studied analog extensions of L_1-regularization techniques, that have demonstrated their adaptiveness. For this two cases, we will present representer theorems, specifying the form of the solutions of the underlying optimization problem, allowing for algorithmic treatments of the optimization tasks.

Harmonic Equiangular Tight Frames Comprised of Regular Simplices

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An equiangular tight frame (ETF) is a sequence of unit-norm vectors in a Euclidean space whose coherence achieves equality in the Welch bound, and thus yields an optimal packing in a projective space. A regular simplex is a simple type of ETF in which the number of vectors is one more than the dimension of the underlying space. More sophisticated examples include harmonic ETFs which equate to difference sets in finite abelian groups. Recently, it was shown that some harmonic ETFs are comprised of regular simplices. We characterize when the subspaces that are spanned by the ETF's regular simplices form an equi-isoclinic tight fusion frame, which is a type of optimal packing in a Grassmannian space. We also provide explicit infinite families of such ETFs.

Convergence Analysis of the ADI Extrapolated Crank–Nicolson Orthogonal Spline Collocation Scheme for Burgers' Equation in 2 Space Variables

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Accurate and efficient numerical schemes for solution of the Navier–Stokes equations can be constructed by considering a pressure Poisson equation reformulation. Applying an alternating direction

implicit extrapolated Crank-Nicolson time discretization to this reformulation breaks the problem in to two parts: Burgers' equation, and Poisson's equation with Neumann boundary conditions. We have previously demonstrated (via numerical experiment) that the orthogonal spline collocation discretization of this scheme exhibits optimal order convergence rates for both the velocity and pressure terms appearing in the Navier–Stokes equations. In this talk, we focus on some of the theoretical justifications of this observation. In particular, we use alternating direction implicit extrapolated Crank-Nicolson orthogonal spline collocation with splines of order r to solve the coupled Burgers' equations in two space variables and two unknown functions. The scheme is initialized with an alternating direction implicit predictor-corrector method. We show theoretically that the H^1 norm of the error at each time level is of order r in space and of order 2 in time. Once again, our numerical results confirm these theoretical orders.

Determining Projection Constants of Univariate Polynomial Spaces

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The long-standing problem of minimal projections is addressed from a computational point of view. Techniques to determine bounds on the projection constants of univariate polynomial spaces are presented. The upper bound, produced by a linear program, and the lower bound, produced by a semidefinite program exploiting the method of moments, are often close enough to deduce the projection constant with reasonable accuracy. The implementation of these programs makes it possible to find the projection constant of several three-dimensional spaces with five digits of accuracy, as well as the projection constants of the spaces of cubic, quartic, and quintic polynomials with four digits of accuracy. Beliefs about uniqueness and shape-preservation of minimal projections are contested along the way.

Optimality of Quasi-interpolation: A Stochastic Perspective

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Probabilistic numerics that aims to study numerical algorithms from a stochastic perspective has now been a surging interdisciplinary field (between numerical approximation and probability) attracting lots of people's attention. Motivated by this driving force, we study the optimality property of quasi-interpolation from a stochastic perspective. We first construct a general quasi-interpolation scheme in a rational form and then derive its optimality in terms of certain quadratic functional in a Hilbert space from both the kernel regression perspective and the perspective of weighted approximation with varying weights. The optimality property shows that quasi-interpolation is simple, robust, and powerful in terms of availability, stability, and generalization ability.

Quasi-interpolation with Cubic Powell–Sabin Splines

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The construction of bivariate polynomial splines on general triangulations can be simplified by applying a Powell–Sabin refinement. Recently, C^1 cubic splines on Powell–Sabin triangulations with

and without additional smoothness constraints have been considered for possible use in approximation theory and geometric modeling. Also, a stable B-form of such splines, which is based on local basis functions that form a convex partition of unity, has been provided. In this talk we make use of the cubic Powell–Sabin B-form to introduce a general framework of methods for constructing quasi-interpolation operators based on local polynomial approximation. We assign a linear functional to each basis function to specify the coefficients in the B-form. These functionals take a cubic polynomial and evaluate its blossom at certain points in the domain. With this approach a quasi-interpolation operator can be defined by providing any cubic approximation method based on local data sites. We study properties of such operators and present general recipes to specify them in a way that the resulting splines satisfy certain additional smoothness constraints. Finally, we derive some concrete methods and compare them with numerical experiments.

Meshless Methods for Manifolds: GMLS Approximations of Hydrodynamic Responses in Curved Fluid Interfaces

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We discuss recent advances in the development of meshless methods for solving partial differential equations on general manifolds. We present a discretization framework based on Generalized Moving Least Squares (GMLS) and exterior calculus formulation of surface PDEs. Motivated by applications arising in soft condensed matter physics and biophysics, we show how our approaches can be used to solve hydrodynamic equations on curved fluid interfaces with spherical topology. We also present convergence results comparing our GMLS methods with a spectral solver in the case of radial manifolds. We then show some advantages of our GMLS methods demonstrating the capability to handle quite general manifold topologies and to adapt numerical resolution. We conclude by showing our recent work in developing a mimetic high-order staggered extension of our aforementioned results to compact topologies.

Nonlinear and Anisotropic Approximation with Gaussian Mixtures

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The *dictionary of Gaussian mixtures* consists of the functions created by all possible affine change of variables of a single Gaussian in $\mathbb{R}^d$. The dictionary is used comprehensively in engineering and scientific applications to a degree that practitioners often use it as their default choice for representing their scientific object. Its pervasive use hinges on the perception that this dictionary is *universal* in the sense that it is large enough and its members are local enough in space and frequency to provide efficient approximation to "almost all objects of interest". However, and perhaps surprisingly, only a handful of concrete theoretical results are actually known on the ability to use Gaussian mixtures in lieu of mainstream representation systems. In this talk I'll introduce a program for nonlinear approximation with Gaussian mixtures and present two concrete results. The first, considered with Amos Ron, features approximation by Gaussians at multiple, spatially varying scales, and provides correct rates for functions in standard smoothness spaces for nonlinear approximation. The second, recently considered with Amos Ron and Wolfgang Erb, treats N-term Gaussian approximation of bivariate functions with anisotropic smoothness, with rates matching those of curvelets and shearlets.

Nonlinear Approximation and Deep ReLU Networks

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Neural networks are widely used for non-linear function approximation in modern machine learning tasks. Their success indicates that there are large classes of functions which can be efficiently captured by neural networks where classical nonlinear methods fall short of the task. I will present several results, joint with Ingrid Daubechies, Ron DeVore, Simon Foucart, and Geurgana Petrova, showing that this is indeed the case.

Linear Programming Bounds for Packing and Energy on the Sphere

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The problem of finding point configurations on a sphere that minimize a repulsive energy functional appears in a number of guises including best-packing problems, coding theory, classical ground state problems, and node distribution. For a potential h defined on $[-1, 1)$ and a finite point configuration (or *code*) C on the unit sphere $\mathbb{S}^{n-1}$, the h-energy of C is given by $E(C, n, h) := \sum_{x \neq y \in C} h(\langle x, y \rangle)$. H. Cohn and A. Kumar proved that every sharp spherical code (a code with m distinct inner products that is also a spherical design of strength $2m - 1$) is universally optimal; i.e., such a code minimizes the potential h-energy among all codes of cardinality $|C|$ for any absolutely monotone potential h.

In this talk, I will review classical linear programming bounds for spherical codes and present new 'universal' lower bounds for energy of the form $E(C, n, h) \geq N^2 \sum_{i=1}^{m} \rho_i h(\alpha_i)$, where the nodes $\{\alpha_i\}$ and weights $\{\rho_i\}$ depend only on the cardinality N and dimension n and are obtained from a quadrature rule framework developed by Levenshtein in relation to maximal codes. These bounds coincide with those of Cohn and Kumar for the case of sharp codes. This is joint work with P. Boyvalenkov, P. Dragnev, E. Saff and M. Stoyanova.

Frozen Gaussian Approximation for High Frequency Elastic Waves

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The frozen Gaussian approximation (FGA) is an effective tool for modeling high frequency wave propagation. In previous works, the convergence of the FGA has established for strict hyperbolic systems using an eigenvector decomposition. With a weak asymptotic expansion and projecting onto a moving frame we methodically derive the evolution equations for the FGA for the elastic wave equation. A diabatic coupling is observed for the amplitude of the evolution equations between the SH, SV waves. Using previous results with energy estimates we establish the convergence for the first order FGA for the elastic wave equation. As a consequence, we show the formulation using the derived evolution equations via projection onto an orthonormal frame is asymptotically equivalent to the eigenvector decomposition of the hyperbolic system.

Fast Cross-validation in Harmonic Approximation

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We are concerned with determining good regularization parameters for Tikhonov regularization problems. To this end we aim at fast algorithms for computing leave-one-out cross-validation scores. In our talk we present a general approach to shift the main computations from the function in question to the node distribution and, making use of FFT and FFT-like algorithms, even reduce this cost tremendously to the cost of solving a single Tikhonov regularization problem. We apply this technique in different settings on the torus, the unit interval, the two-dimensional sphere and the rotation group. Given that the sampling points satisfy a quadrature rule our algorithm computes the cross-validations scores in floating-point precision. In the cases of arbitrarily scattered nodes we propose an approximating algorithm with the same complexity. Numerical experiments indicate the applicability of our algorithms.

CUR Decompositions and Perturbations

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This talk will discuss a useful tool in dimensionality reduction and low-rank matrix approximation called the CUR decomposition which decomposes a matrix by selecting representative columns and rows from it. We would present several equivalent formulations for CUR decomposition, and discuss a randomized row/column selection method to guarantee the exact CUR decomposition of a low-rank matrix with high probability. Additionally, a novel perturbation analysis is performed on CUR approximations of noisy versions of low-rank matrices, which compares them with the putative CUR decomposition of the underlying low-rank part. And we also provide numerical illustrations of the methods and bounds discussed.

Frames and Numerical Approximation

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Spectral and high order function approximation can be difficult for non-smooth functions, or for functions defined on complicated domains. We show that such difficulties can be lifted by considering approximations using a frame, rather than a basis. Frames are more flexible than a basis, and in particular they can be redundant. This allows incorporating known features of the function to be approximated into the approximation space. These features may be jumps, other point discontinuities or singularities, known oscillatory behaviour, ... We consider the numerical stability of function approximation using frames, as well as efficient algorithms for computing the approximation. The main method is the AZ algorithm, which in a generic way exploits the redundancy that is inherent in several example frames.

Phase Retrieval from Windowed Fourier Measurements with Associated Lower Bounds

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We will discuss phase retrieval from locally supported STFT magnitude measurements of a vector $\mathbf{x}$ based on a two-step approach: First, a modified Wigner Distribution Deconvolution (WDD) approach is used to solve for a portion of the lifted rank-one signal $\mathbf{x}\mathbf{x}^*$. Second, an angular synchronization approach is used to recover $\mathbf{x}$ up to a global phase factor from the known portion of $\mathbf{x}\mathbf{x}^*$. We will also discuss lower bounds for the Lipschitz continuity of these measurements based off of the size of the support of our local measurement masks. These lower bounds are independent of our reconstruction algorithm and so give insight into the best possible performance of any such method.

Equiangular Tight Frames from Group Divisible Designs

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Several applications in signal processing require lines through the origin of a finite-dimensional Hilbert space with the property that the smallest interior angle is as large as possible. One class of optimal packings, those that achieve equality in the Welch bound, are known as equiangular tight frames (ETFs). The central problem in the study of ETFs is the question of existence, that is, given integers n and d, does there exist an ETF with n lines in d-dimensional space? To tackle this problem the primary approach has been to develop new constructions of ETFs. In this talk we will present a new construction of ETFs using a combinatorial object known as a group divisible design. We will see that this new construction not only provides a new infinite family of ETFs but it also sheds some light on the question of existence for general n and d.

Offline-enhanced Reduced Basis Method through Adaptive Construction of the Surrogate Training Set

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The Reduced Basis Method (RBM) is a popular certified model reduction approach for solving parametrized partial differential equations. One critical stage of the offline portion of the algorithm is a greedy algorithm, requiring maximization of an error estimate over parameter space. In practice this maximization is usually performed by replacing the parameter domain continuum with a discrete "training" set. When the dimension of parameter space is large, it is necessary to significantly increase the size of this training set in order to effectively search parameter space. Large training sets diminish the attractiveness of RBM algorithms since this proportionally increases the cost of the offline phase.

In this work we propose novel strategies for offline RBM algorithms that mitigate the computational difficulty of maximizing error estimates over a training set. The main idea is to identify a subset of the training set, a Surrogate Training Set(STS), on which to perform greedy algorithms. The STSs we construct are much smaller in size than the full training set, yet our examples suggest that they are accurate enough to induce the solution manifold of interest at the current offline RBM iteration. We propose two algorithms to construct the STS: Our first algorithm, the Successive Maximization Method

(SMM) method, is inspired by inverse transform sampling for non-standard univariate probability distributions. The second constructs an STS by identifying pivots in the Cholesky Decomposition of an approximate error correlation matrix. We demonstrate the algorithm through numerical experiments, showing that it is capable of accelerating offline RBM procedures without degrading accuracy, assuming that the solution manifold has rapidly decaying Kolmogorov width.

An FFT Approach for High Dimensional Approximation using Multiple Rank-1 Lattices

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The talk illustrates an efficient construction method for spatial discretizations of multivariate trigonometric polynomials $p(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in I} \hat{p}_{\boldsymbol{k}} \mathrm{e}^{2\pi \mathrm{i} \boldsymbol{k}\cdot\boldsymbol{x}}$ with (almost) arbitrary frequency support $I \subset \mathbb{Z}^d$, $|I| < \infty$, using a set of rank-1 lattices. The structure of the resulting sampling schemes allow for efficient evaluation as well as reconstruction algorithms, i.e., fast Fourier transform algorithms, with a complexity in $\mathcal{O}(M \log M + |I|(d + \log |I|) \log |I|)$, where $|I|$ is the number of frequencies, $M \lesssim |I| \log |I|$ is the number of sampling nodes within the sampling scheme, and d is the spatial dimension. The aforementioned algorithms can be directly applied in order to approximate multivariate periodic functions that belong to reproducing kernel Hilbert spaces, e.g., function spaces of dominating mixed smoothness. This leads to an approximation method, that yields nearly optimal L_∞ worst case approximation errors even in terms of the number of used sampling values. Another major advantage of the new sampling method is the preferable pre-asymptotic behavior of its worst case sampling errors.

Computational Methods for Kernel-Based Cubature

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For N data points, the naive implementation of Bayesian cubature and other kernel-based methods is based on solving a linear system of N equations. The resulting cubic computational and quadratic memory cost in N restrict the applicability of these methods. We show how relatively flexible *fully symmetric sets*, obtained from given vectors via coordinate permutations and sign-changes, can be exploited for efficient computation of the weights of kernel-based cubature rules for up to tens of millions of points. If the point set is a union of J fully symmetric sets, computational complexity is reduced from $\mathcal{O}(N^3)$ to $\mathcal{O}(J^3 + JN)$ and memory complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(J^2)$. In its current state the algorithm does not adapt to efficient optimization of kernel hyperparameters, which is a crucial step in uncertainty calibration of a Bayesian cubature method. We also briefly discuss some other recent approaches based on sparse grids and a combination of low discrepancy points and shift-invariant kernels.

Reconstruction of Non-Stationary Signals by the Generalized Prony Method

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In 2013 Peter & Plonka derived a generalized Prony method for the reconstruction of sparse expansions of eigenfunctions of suitable linear operators using only a small number of suitable sample values.

Although some examples were given there, the problem of finding such operators is non-trivial. In this talk we consider different generalizations of shift operators and their corresponding sets of eigenfunctions and eigenvalues that admit a reconstruction of structured functions from function values. In particular, we present a way to reconstruct signal such as arbitrary linear combinations of shifted Gaussians, Gabor expansions with Gaussian windows and non-stationary trigonometric expansions with special monotone phase functions via the generalized Prony method.

Approximation of Multivariate Functions on Sparse Grids by Quasi-interpolation Based on Radial Basis Functions

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In this talk, we present a new family of quasi-interpolation schemes for the approximation of multivariate functions on sparse grids. The sparse grids are a discrete blend of directionally uniform grids, and the basis functions used in our quasi-interpolants are anisotropic extensions of (compact and nearly-compact) bell-shaped kernels constructed from radial basis functions (e.g., multiquadrics). The main advantages of our quasi-interpolants are two-fold: they provide optimal rates of approximation without the need to solve linear systems, and they dramatically reduce the amount of data required. We implement our scheme using both single-level and multilevel paradigms designed to improve rates of approximation. We provide a rigorous proof for the single-level approximation orders on sparse grids and present numerical results to demonstrate the performance of the proposed scheme. In particular, we show that our quasi-interpolants provide better rates of approximation than a comparable method in the literature based on the Gaussian kernel.

Convergence Rates of Gaussian ODE Filters

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A recently-introduced class of probabilistic (uncertainty-aware) solvers for ordinary differential equations (ODEs) applies Gaussian (Kalman) filtering to initial value problems. These methods model the true solution x and its first q derivatives *a priori* as a Gauss–Markov process $\boldsymbol{X}$, which is then iteratively conditioned on information about $\dot{x}$. We prove worst-case local convergence rates of order h^{q+1} for a wide range of versions of this Gaussian ODE filter, as well as global convergence rates of order h^q in the case of $q = 1$ and an integrated Brownian motion prior, and analyse how inaccurate information on $\dot{x}$ coming from approximate evaluations of f affects these rates. Moreover, we present explicit formulas for the steady states and show that the posterior credible intervals are well calibrated in all considered cases that exhibit global convergence—in the sense that they globally contract at the same rate as the truncation error.

Time-variant System Approximation via Later-time Samples

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Let $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ be frames for $L_2([0,1])$ and $L_2([0,\infty))$, respectively. We consider a function $u(x,t)$ which can be represented by frame elements as following: $u(x,t) = \sum_{n=1}^{\infty} a_n f_n(x) g_n(t)$,

where $a_n \in \mathbb{R}$, $x \in [0,1]$, and $t \in [0,\infty)$. We explore when the initial datum $f(x) = u(x,0)$ can be approximated in a reasonable sense from the given information $u(x_0,t_1), u(x_0,t_2), \ldots, u(x_N,t_N)$.

A Prony-Laplace Method for Identifying Burst-like Forcing Terms

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We consider the problem of recovery of a burst-like forcing term f in the initial value problem

$$\dot{x} = Ax + f, \quad x(0) = 0,$$

in the framework of dynamical sampling. We introduce the notion of a sensing limit of a collection of samples with respect to the semigroup generated by A and indicate its fundamental role in the solvability of the problem. We also show that in certain cases a combination of Laplace transform and Prony's methods can be used for identifying the forcing term.

High-dimensional Integration and Approximation: The Quasi-Monte Carlo (QMC) Way

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High dimensional computation – that is, numerical computation in which there are very many or even infinitely many continuous variables – is a new frontier in scientific computing. Often the high dimensionality comes from uncertainty or randomness in the model or data (e.g., in groundwater flow it can arise from modeling the permeability field that is rapidly varying and uncertain). High dimensional problems present major challenges to computational resources, and require serious theoretical numerical analysis in devising new and effective methods.

This talk will begin with a contemporary review of Quasi-Monte Carlo (QMC) methods, which offer tailored point constructions for solving high dimensional integration and approximation problems by sampling. By exploiting the smoothness properties of the underlying mathematical functions, QMC methods are proven to achieve higher order convergence rates, beating standard Monte Carlo sampling. Moreover, QMC error bounds can be independent of the dimension under appropriate theoretical function space settings.

In recent years the modern QMC theory has been successfully applied to a number of applications in uncertainty quantification. This talk will showcase some ongoing works where we take QMC methods to new territories including neutron transport as a high dimensional PDE eigenvalue problem, high frequency wave scattering in random media, optimal control constrained by PDE with random coefficients, and a revolutionary approach to model random fields using periodic random variables. The common and essential theme among these collaborations with various international teams is that we are not just applying an off-the-shelf method; rather, we provide rigorous error and cost analysis to design QMC methods tailored to the features of the underlying mathematical functions in these applications.

Localized Meshless Methods for Solving Telegraph Equations

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We present a newly developed localized meshless method using radial basis functions and apply it to solving the telegraph equation. The method uses small neighborhoods of points to find the approximate solution of the given partial differential equation, and requires only the solution of a large sparse matrix. We also extend the localized meshless method to solving time-dependent problems with higher order time-marching schemes, and introduce adaptive approaches that allows the solution of large-scale engineering problems.

On DC Based Methods for Phase Retrieval

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We explain a new computational approach to solve a broader class of phase retrieval problems. The approach splits a standard nonlinear least squares minimizing function associated with the phase retrieval problem into the difference of two convex functions and then solves a sequence of convex minimization sub-problems. For each subproblem, the Nestrov accelerated gradient descent algorithm or the Barzilai-Borwein (BB) algorithm is used. In the setting of sparse phase retrieval, a standard ℓ_1 norm term is added into the minimization mentioned above. The subproblem is approximated by a proximal gradient method which is solved by the shrinkage-threshold technique directly without iterations. In addition, a modified Attouch-Peypouquet technique is used to accelerate the iterative computation. These lead to more effective algorithms than the Wirtinger flow (WF) algorithm and the Gauss-Newton (GN) algorithm and etc.. A convergence analysis of both DC based algorithms shows that the iterative solutions is convergent linearly to a critical point and will be closer to a global minimizer than the given initial starting point. Our study is a deterministic analysis while the study for the Wirtinger flow (WF) algorithm and its variants, the Gauss-Newton (GN) algorithm, the trust region algorithm is based on the probability analysis. In particular, the DC based algorithms are able to retrieve solutions using a number m of measurements which is about twice of the number n of entries in the solution with high frequency of successes. When $m \approx n$, the ℓ_1 DC based algorithm is able to retrieve sparse signals. Numerical simulations will be shown at the end of the talk.

Localized Radial Basis Function Methods for PDEs in Thin Volumes

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When simulating the biomechanical function of the respiratory system, the most important muscle is the diaphragm. It spans the width of the human thorax region, but has a thickness that is about a hundred times smaller than the width. To create a three-dimensional simulation of the muscle action, the thickness dimension needs to be resolved. In this talk, we discuss how to represent the geometry of the diaphragm, which is extracted from medical images. We discuss what types of node sets to generate in the volume. In particular, we focus on anisotropic node sets as these reduce the overall computational cost significantly. Then we analyze the properties of anisotropic polyharmonic spline approximations, and under which conditions these can be more accurate than isotropic approximations. Finally, results for a simplified linear elasticity problem are shown.

The Construction of Multivariate Spline Interpolation Bases for FEM

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As we know, the construction of the interpolation bases or the shape functions is a key problem of the finite element method. The polynomial interpolation bases are usually applied in the traditional FEM. However, due to the limitation of polynomial interpolation method, we can only construct interpolation bases for triangular, rectangular, tetrahedral or regular prism elements directly. For some irregular elements, e.g., quadrilateral element, we have to use isoparametric transformation. For polygonal element, rational functions are needed. In general, the shape functions cannot have high order completeness or high accuracy for distorted meshes. In this talk, we introduce the construction of multivariate spline interpolation bases for some irregular elements, which can possess high order completeness.

Topology Preserving Approximation for Aircraft Conceptual Design

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This presentation highlights the mathematical challenges involved in automated generation of computational fluid dynamic (CFD) and finite-element (FEM) meshes. A relevant topology preserving approximation problem is formulated to obtain an exact boundary representation of a geometry model for automated mesh generation. Topology preserving approximations can enable automated CFD and FEM analyses for aircraft conceptual design as demonstrated by two application examples.

Marcinkiewicz Inequalities for Hermite-Biehler Weights

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The classical Marcinkiewicz inequality compares discrete and integral p-norms in spaces of trigonometric polynomials. This talk discusses analogues of these inequalities for entire functions of exponential type in weighted L^p-spaces ($1 < p < \infty$). The weights are assumed to be of the form $|E(x)|^{-p}$ where E is a Hermite-Biehler entire function. A consequence of these results are convergence statements for Lagrange interpolation series in weighted spaces. As a special case, these inequalities recover results of Grozev and Rahman and of Lubinsky on Marcinkiewicz inequalities for entire functions in L^p-spaces with homogeneous weights $|x|^{(\alpha+\frac{1}{2})p}$.

Distribution of Eigenvalues of Toeplitz Matrices with Smooth Entries

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Determinants of Toeplitz matrices play a key role in areas ranging from Padé approximation to orthogonal polynomials. In the case of Padé approximation, the entries of the Toeplitz matrix are the Maclaurin series coefficients of a function analytic in a disc containing 0. We establish that when the coefficients are smooth in a certain sense, then there is an explicit asymptotic distribution of the eigenvalues as the matrix size grows.

Recovering Low-rank Matrices from Binary Measurements

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We will consider the approximate recovery of low-rank matrices acquired through binary measurements. Two types of recovery algorithms are considered, one based on hard singular value thresholding and the other one based on semidefinite programming. When no thresholds are used before binary quantization, the direction of the low-rank matrices can be well approximated. Furthermore, the magnitude can also be recovered by introducing thresholds and choosing these thresholds adaptively leads to exponential decay in the approximation error. The arguments are essentially deterministic, relying only on an unusual restricted isometry property of the measurement process.

Gabor Frames on Model Sets

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Model sets were introduced by Meyer in his study of harmonious sets and they are non-uniform sets that are a very natural generalizations of lattices. In this presentation we will consider Gabor frames on such irregular sets. We generalize main concepts of Gabor analysis to the setting of model sets. We construct Janssen's representation of the frame operator and Wexler-Raz biorthogonality relations utilizing the connection between model sets and almost periodic functions, as well as Poisson's summations formula for model sets.

Adaptive Approximation by Optimal Weighted Least-squares Methods

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We discuss adaptive approximation of a multivariate real-valued function ϕ in arbitrary dimension, by means of optimal weighted least-squares estimators. Each estimator is constructed from pointwise evaluations of ϕ at random samples. When the random samples follow a suitable probability distribution, a stable and accurate adaptive estimator of ϕ can be constructed by using a number of samples only linearly proportional (up to log terms) to the dimension of the approximation space.

Data-dependent Distances for Unsupervised Learning

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Approaches to unsupervised clustering with data-dependent distances are proposed. By considering metrics derived from data-driven graphs, robustness to noise and cluster geometry is achieved. The proposed algorithms enjoy theoretical guarantees on flexible data models, and also have quasilinear computational complexity in the number of data points. Applications to a range of real data will be

shown, demonstrating the practical applicability of our methods. *Joint with Anna Little (Michigan State University) and Mauro Maggioni (Johns Hopkins University).*

Least-squares Approximation with General Distributional Data

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We consider the problem of approximating functions in Sobolev spaces from randomly sampled measurements using a discrete least squares procedure. Our setup allows for general, vector-valued measurements whose components may be function derivative values of arbitrary order. We show that, given a particular class of measurement data, there is an optimal sampling strategy if independent and identically distributed measurements are collected. This optimal sampling strategy ensures that the computed approximation is proximal to the best approximation from a given Sobolev space that is consistent with the collected measurements. We demonstrate on some numerical examples that the optimal sampling strategy can be implemented with a practical algorithm in high-dimensional spaces. Our theoretical results include new bounds on the error of the approximation, provide error estimates in Sobolev spaces, and consider general cases when heterogeneous measurement types are gathered.

Enhanced Meshfree Near-boundary Approximation via Extrapolation

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In this talk we will discuss enhancing near-boundary meshfree approximation with localized bases constructed from thin-plate spline kernels or Matérn kernels. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and let $X \subset \Omega$ be finite and quasi uniform. Standard scattered-data meshfree approximations work well in the *interior* of Ω, but they degrade in the vicinity of the boundary. The reason is that when a point x is near the boundary, the centers in a ball about x are no longer quasi-uniform in the whole ball, just in the half farthest from $\partial\Omega$. A particular near-boundary approximation problem comes up in the meshfree particle methods associated with the Babuska-Banerjee-Osborn (t, k) regular systems. To deal with it, they needed a Sobolev extension of the underlying function; however, obtaining such an extension is computationally problematic, given only values of a function on $X \subset \Omega$. Here, we introduce a novel meshfree method that ameliorates such problems by extrapolating the function values known on X to values at centers in the *exterior* of Ω. The extrapolation method yields rates of approximation near $\partial\Omega$ that are comparable to those in the interior, up to logarithmic factors.

Simple approaches to complicated data analysis

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Recent advances in technology have led to a monumental increase in large-scale data across many platforms. One mathematical model that has gained a lot of recent attention is the use of sparsity. Sparsity captures the idea that high dimensional signals often contain a very small amount of intrinsic information. Using this notion, one may design efficient low-dimensional representations of large-scale

data as well as robust reconstruction methods for those representations. Binary, or one-bit, representations of data for example, arise naturally in many applications, and are appealing in both hardware implementations and algorithm design. In this talk, we provide a brief background to sparsity and 1-bit measurements, and present new results on the problem of data classification with low computation and resource costs. We illustrate the utility of the proposed approach on recently acquired data about Lyme disease.

Point-exchanging Methods for Obtaining Kernel Quadrature Formula

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For high dimensional integration, many kinds of kernel quadrature are considered. They generate sample points by adding points greedily so that the worst case error of numerical integration is minimized. In this talk, we present alternative ways to obtain good sample points by improving the points sets without adding points. Our methods improve randomly generated sample points and we have proved that the worst case error is $\mathrm{O}\left(n^{-\frac{1}{2}}\right)$ where n is the number of sample points. This order is consistent with the numerical experiments.

Nonlinear Approximation by Function Composition with Multi-Layer Neural Networks

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For Hölder continuous functions of order $\alpha > 0$ and constant $\nu > 0$ defined on a $d-$dimensional cube, $N-$term nonlinear approximation by function composition can, following recent results by Shen et al., achieve the rate of approximation $O(\nu N^{-2\alpha/d})$ with feed-forward neural networks (FFNN) comprising $O(1)$ hidden layers. Since the $L-$function composition is governed by the activation function σ and the $L-$hidden layer structure of the FFNN, this technique enables to formulate their approximation capacity with respect to the number of layers (depth) and number of units per layer (width) –instead of quantifying this property asymptotically in function of the total number of trainable parameters. Following the characterization of the rate of nonlinear approximation by certain smoothness conditions required on the approximated function f, we aim at extending this constructive method for the approximation of functions belonging to the Sobolev and Besov spaces. Then, given the properties of the target function f and an approximation error ϵ, one can tune the FFNN structure to balance the tradeoff between their expressivity and trainability.

Old and New Bounds for Projective Cubature Formulas

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Papers of Nozaki and Sawa from 2012 and Lyubich and Shatalova from 2013 collected upper bounds for minimal sized projective cubature formulas of even index. Here it is shown these bounds can be improved in some cases. Some discussion on these formulas in relation to energy minimization in projective spaces will be described, along with the methods used to find explicit small sized cubature formulas.

A Quasi-interpolation Method Based on LR B-splines

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In order to break down the tensor structure of standard B-splines, Locally Refined (LR) B-splines have been introduced by Dokken T. et al. in 2013 by extending the concept of knot insertion of the 1D B-splines to local insertion of (d-1)-dimensional boxes for the dD case. Like standard tensor-product B-splines, LR B-splines have local supports, are nonnegative and, using weights in (0,1), they form a partition of unity. However, a full description of their linear independence is still an open problem. LR B-splines are defined over mesh instances, called LR-meshes. These are built as a sequence of local insertions starting from a coarse tensor mesh. In 2015, Bressan A. and Jüttler B. provided a way of generating LR-meshes over which the corresponding LR B-splines are locally linearly independent. These meshes have a hierarchical structure, and the procedure requires an a priori knowledge of the subregions of the domain where the mesh should be finer.

On the other hand, a quick and light construction of quasi-interpolation schemes based on Truncated Hierarchical (TH) B-splines has been developed by Speleers H. and Manni C. in 2016. It is actually also applicable in the general setting where the basis functions have local supports, are nonnegative, form a partition of unity and are locally linearly independent. Moreover, it is proved that such a quasi-interpolant is actually a projector on the space spanned by the basis functions under some not-so-restrictive hypotheses on the given data set. In this talk we combine the above results to get a quasi-interpolant based on LR B-splines. We provide some numerical examples, and make comparisons with THB-splines and tensor-product B-splines.

Advances in the Approximation Theory of Generalized Moving Least Squares

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In this talk we present existence and approximation results for the reconstruction of a few classes of linear functionals, including differential and integral functionals, using the Generalized Moving Least Square (GMLS) method. These results extend or specialize classical MLS theoretical results, and they rely both on the classic approximation theory for finite elements and on existence/approximation results for scattered data. In particular, we will consider the reconstruction of vector fields in Sobolev spaces and the reconstruction of differential k-forms. We show how these results can be applied to data transfer problems and to design collocation and variational meshless schemes for the solution of partial differential equations.

Approximation of High-dimensional PDEs by Neural Networks

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We present approximation theoretical results where neural networks overcome the curse of dimension when approximating high-dimensional functions. We focus on functions that are the solutions of (parametric) PDEs. Additionally, we analyse the structure of spaces of neural networks to demonstrate that these approximation systems are quite different from traditional systems, such as, polynomials, splines, or wavelets.

Rearranged Fourier Series and Generalizations to Non-commutative Groups

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Fourier series of continuous functions are not always uniformly convergent. However, P. L. Ulyanov proposed a problem: can we permute the Fourier series of each individual continuous function in such a way as to guarantee uniform convergence of the rearranged Fourier series? This problem remains open, but nonetheless a rather strong partial result was proved by S. G. Revész which states that for every continuous function there exists a subsequence of rearranged partial Fourier sums converging to the function uniformly. We give several new equivalences to Ulyanov's problem in terms of the convergence of the rearranged Fourier series in the strong and weak operator topologies on the space of bounded operators on $L_2(\mathbb{T})$. This new approach gives rise to several new problems related to rearrangement of Fourier series. We also consider Ulyanov's problem and Revész theorem for reduced C^*–algebras on discrete countable groups.

Quasi-interpolants and the Solution of Fractional Differential Problems

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In recent years fractional calculus has proved to be a powerful tool to describe real-world phenomena. Differential problems of fractional, i.e. noninteger, order are widely used in several fields, from physics to continuum mechanics, from signal processing to electromagnetism. The analytical solution of fractional differential problems is given just in same special cases and, usually, is expressed in term of special functions or series expansions. For this reason, the construction of efficient numerical methods to solve fractional differential problems is of great interest. In this talk we present a numerical method, based on quasi-interpolant operators, suitable to solve time-fractional differential problems. We analyze its approximation properties and show its performance in solving some test problems. This is a joint work with E. Pellegrino and L. Pezza.

Node Generation for High-order Approximation

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Spectral and high-order methods are known to be sensitive to node placement. The classical example being polynomial interpolation, which is exponentially ill-conditioned unless nodes are clustered near the boundary of the approximation domain. Although sensitivity to node placement can be mitigated by using more localized approximation schemes, such as radial basis functions or high-order finite differences, instabilities can still arise due to poor point distribution. In this talk we review known procedures to generate points for polynomial approximation and extend them to more general settings, including constrained approximation, frame approximation, and radial basis function methods. The use of such nodes for the solution of PDEs on general geometries will also be discussed.

On the Impact of Prony's Method

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The recovery of signals which can be represented or approximated by finite expansions into signal atoms is a task regularly encountered in a variety of fields such as signal processing, biology, and engineering. In some situations the atoms of these expansions do not even form a finite or countable basis or a frame but can be taken from an uncountable set of parametric functions. These "function atoms" have a fixed structure and can be identified by a small number of real or complex parameters. The most prominent and well-studied signal model is a sparse expansion into complex exponentials, i.e.,

$$f(x) := \sum_{j=1}^{M} c_j \exp(T_j x) = \sum_{j=1}^{M} c_j z_j^x, \tag{1}$$

with pairwise different $z_j := \exp(T_j)$ and with parameters $c_j \in \mathbb{C} \setminus \{0\}$ and $T_j \in \mathbb{C}$. Using the classical Prony method, the parameters c_j and z_j can be theoretically computed from the $2M$ equidistant samples $f(\ell)$, $\ell = 0, \ldots, 2M-1$. In practical applications however, we have to take special care of the numerical instabilities that can occur using Prony's method.

The interest in Prony-like methods has been strongly increased during the last years, also because of their utilization for the recovery of signals of finite rate if innovation. In particular, the close connection between the exponential sum in (1) and the expansion into shifted Diracs $s(t) = \sum_{j=1}^{M} c_j \, \delta(t - t_j)$ with $c_j \in \mathbb{C} \setminus \{0\}$ and $t_j \in \mathbb{R}$ is extensively used. Indeed the Fourier transform of $s(t)$ is of the form (1), where $T_j = \mathrm{i}t_j$, and thus $s(t)$ can be reconstructed from only $2M$ of its Fourier samples.

There are close connections between Prony's method and several other mathematical tools as Padé approximation, annihilating filter method and linear prediction. Modified Prony methods relate to low-rank approximation problems and sublinear sparse FFT algorithms. The Prony method can be generalized for the recovery of sparse expansions into eigenfunctions of linear operators. The talk gives an overview about some new results on Prony-like methods.

The Cone-beam Transform and Spherical Convolutions

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The cone-beam tomography consists of integrating a function f that is defined on the three-dimensional space $\mathbb{R}^3$ along every ray that starts on a certain scanning set. The task is to find a good approximation of f given these integrals. Based on Grangeat's formula, [Louis 2016, Inverse Problems] states reconstruction formulas for the cone-beam tomography based on a generalized Funk–Radon transform on the sphere. In this talk, we give a singular value decomposition of this generalized Funk–Radon transform and apply this result to derive a singular value decomposition of the cone-beam transform with sources on the sphere $\mathbb{S}^2$.

Stochastic Collocation with Hierarchical Extended B-splines on Spatially Adaptive Sparse Grids

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B-splines are widely used for the approximation of smooth functions. However, the classical tensor product approach based on uniform isotropic grids reaches its limits in about four dimensions. That is because the number of grid points increases exponentially with the dimensionality of the parameter space. Sparse grids are an established technique to tackle this curse of dimensionality. In previous work we introduced spatial adaptivity, which automatically selects only the most significant sparse grid points for a given objective function and a new basis of hierarchical B-splines. Further increasing the dimensionality, the sparse grid's boundary points must be omitted too. To compensate for the missing points, the B-spline basis functions are so far modified according to natural boundary conditions. However, modified B-splines do not span the complete polynomial space anymore and therefore lack one of the fundamental spline properties.

In this work we introduce hierarchical extended B-splines for spatially adaptive sparse grids. This approach is based on the polynomial basis property. It is more general and mathematically rigorous than previous boundary treatment for spline functions on sparse grids. We prove the polynomial basis property of the new hierarchical basis and show empirical convergence results. Furthermore we apply the new basis to problems in uncertainty quantification via stochastic collocation. This demonstrates that hierarchical extended B-splines do not only represent polynomial functions exactly, but also improve the approximation quality for general objective functions and quantities of interest.

Optimal Spline Quasi-interpolation on Type-1 Triangulations

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Quasi-interpolation is a general and powerful approximation approach for defining local approximants to a given function or a given set of data with low computational cost. In this talk we present a method for the construction of quasi-interpolating splines where the spline is directly determined by setting its Bernstein-Bézier coefficients to appropriate combinations of the given data values instead of defining it as linear combinations of compactly supported bivariate splines. In particular we consider the space of C^1 quartic and cubic splines on type-1 triangulations and obtain quasi-interpolating operators exact on cubic and quadratic polynomials, respectively. Moreover, in the space of C^1 quartic splines, we propose the construction of interpolating splines, by conveniently modifying the quasi-interpolating ones there constructed. To be more precise, the idea is to use a refinement step of the well know butterfly subdivision scheme for triangular regions, to move from a coarse grid to a finer one, maintaining the same degree of polynomial reproduction. We analyse the properties of the proposed approximants and provide some numerical tests, confirming the theoretical results.

Discrete Energy and Polarization on Fractal Sets

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We discuss the asymptotic behavior of minimal discrete energy and maximal discrete polarization constants for Riesz potentials on fractal sets. When the order of the Riesz potential is big enough, these constants do not behave as in the case of rectifiable set of integer dimension. We will discuss these phenomena and state several positive results when the asymptotic behavior exists.

Kernel Methods for Parametric Equations

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In this talk, I shall discuss recent progress in kernel methods for parametric differential equations. This includes connections to greedy aproximation methods as well as deep kernel based neural networks. I shall report on a theoretical error analysis as well as practical experiences in applications from fluid dynamics.

The Power of Deeper Networks for Expressing Natural Functions

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It is well-known that neural networks are universal approximators, but that deeper networks tend in practice to be more powerful than shallower ones. We shed light on this by proving that the total number of neurons m required to approximate natural classes of multivariate polynomials of n variables grows only linearly with n for deep neural networks, but grows exponentially when merely a single hidden layer is allowed. We also provide evidence that when the number of hidden layers is increased from 1 to k, the neuron requirement grows exponentially not with n but with $n^{1/k}$, suggesting that the minimum number of layers required for practical expressibility grows only logarithmically with n.

An Application of QI–based Quadrature Rules to Isogeometric Boundary Element Methods

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Boundary element methods (BEMs) can be considered in some cases a valid alternative to classical domain methods, such as finite differences and finite elements, to solve partial differential equations. Through the fundamental solution, the original problem can be reformulated by integral equations defined on the boundary of the domain. These methods have two main advantages, the dimension reduction of the computational domain and the simplicity for treating external problems. One of the important challenges in this topic is to accurately and efficiently solve singular integrals that arise from the boundary integral equations so formulated. Therefore, designing suitable quadrature schemes is one of the main active research topic in BEM. With the advent of Isogeometric Analysis (IGA) a new formulation of BEMs has been studied, where the discretization spaces are splines spaces represented in B-spline form. Recently new quasi-interpolation (QI) based quadrature rules have been introduced specifically for IgA-BEM setting. Such quadrature schemes are tailored for B-splines and provide very good accuracy and optimal convergence rate. Weakly, strongly and hyper-singular integrals related to the 2D integral formulation of the Laplace equation with different types of boundary conditions have been then approximated by using these new rules, and have given promising results. Moreover local refinability of the approximated solution of the problem can be also achieved by using hierarchical B-spline spaces. It can be seen that the local nature of the QI perfectly fits with hierarchical spline constructions and leads to an efficient and accurate numerical scheme. The talk is based on results of several joint collaborations with A. Aimi, F. Calabrò, A. Falini, C. Giannelli, T. Kanduč and A. Sestini.

Explicit Constants in Tensor Product Spline Approximations

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In this talk we provide a priori error estimates in standard Sobolev norms for approximation in spline spaces of maximal smoothness on tensor product grids. The error estimates are expressed in terms of a power of the maximal grid spacing, the partial derivatives of the function to be approximated, and an explicit constant. Attention is paid to the periodic case where one can show that the obtained constant is sharp. The results of this talk can be used to theoretically explain the benefits of spline approximation under k-refinement by isogeometric discretization methods. They also form a theoretical foundation for the outperformance of smooth spline discretizations of eigenvalue problems that has been numerically observed in the literature, and for optimality of geometric multigrid solvers in the isogeometric analysis context.

Quadrature Rules Based on Quasi–interpolation for B–spline Weighted Singular and Hypersingular Integrals

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B-spline weighted singular and hypersingular integrals appear in the context of IgA boundary element methods. For their numerical approximation we propose a new class of quadrature rules based on spline quasi–interpolation. In order to define rules active on the whole support of the B–spline weight, we adopt a method based on two basic steps. First, a suited quasi–interpolant is used to approximate the regular part of the integrand and then an algorithm performing the spline product in the B–spline basis is used. The integral of the so approximated integrand can be exactly computed, since a preliminary computation of the modified moments (integrals of products between a fixed singular kernel and B-spline functions) can be done. Convergence results for the case of sufficiently smooth integrands will be introduced, together with comments on the less regular case. This talk is based on joint researches developed with A. Aimi, F. Calabrò, A. Falini, M.L. Sampoli.

Automatic Hyperviscosity-based Stabilization of RBF-FD Discretizations

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Many high-order methods (finite elements, meshless methods) for approximating hyperbolic differential operators typically produce differentiation matrices whose spectra contain spurious eigenvalues with positive real parts (corresponding to spurious eigenvectors), especially on irregular grids/meshes or node sets. These spurious modes lead to explosive instabilities in the context of time-dependent PDEs, necessitating some form of numerical stabilization. To address this issue, we present an automatic hyperviscosity formulation for the stabilization of high-order discretizations of PDEs on both Euclidean domains and manifolds. Our formulation generalizes traditional spectral superviscosity formulations (from Tadmor, Ma, and others) to scattered nodes on both irregular domains and manifolds, without any need for hand-tuning the amount of stabilizing hyperviscosity. This new formulation is quasi-analytic and can be computed using very crude numerical estimates of the spurious eigenvalues on a given node set. In addition, we present the first study (to the best of our knowledge) of the scaling of

spurious eigenvalues in RBF-FD differentiation matrices as a function of mesh norm and approximation order. We use these results to first estimate hyperviscosity coefficients on very coarse node sets, then automatically scale the coarse-node-set coefficients to finer node sets. We show that our formulation helps recover high-order convergence rates on a variety of transport problems over a range of Peclet numbers when used with stagnation-free RBF-FD methods on Euclidean domains and manifolds.

On the Suitability of Neural Networks for the Simulation of Quantum Many-body Systems

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Understanding phenomena in systems of many interacting quantum particles, known as quantum many-body systems, is one of the most sought-after objectives in contemporary physics research. The challenge of simulating such systems lies in the extensive resources required for exactly modeling quantum wave-functions, which grows exponentially with the number of particles. Recently, neural networks were demonstrated to be a promising approximation method of quantum wave functions. However, thus far this approach was mostly focused on more traditional architectures as Restricted Boltzmann Machines and small fully connected networks. In this talk, we propose a method for scaling this approach to support large modern architecture.
We first establish the theoretical potential of such architectures, in the form of deep convolutional and recurrent networks, by proving they can efficiently represent the wave functions of highly entangled quantum systems, which prior methods have struggled with, using polynomially fewer parameters. In contrast to more general works on the expressive efficiency of neural networks, ours is focused on their capacity to represent physical traits. Though significantly more expressive, such architectures do not lend themselves to the conventional methods for employing neural networks for simulating quantum systems. A key part of the simulation is to sample according to the underlying distribution of particle configurations. Current methods rely on Markov-Chain Monte-Carlo sampling, which is too expensive for use with modern architectures, effectively limiting their usable size and capacity. Inspired by recent generative models, we propose a specialized deep convolutional architecture that supports efficient and exact sampling, completely circumventing the need for Markov Chain sampling. We demonstrate our approach can obtain accurate results on larger system sizes than those currently accessible to other neural-network representation of quantum states.

On the Gradient Conjecture and Some Density Problems

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The following "gradient conjecture" was proposed by the Author and Tom McKinley: *Let p and f be homogeneous polynomials in n variables and $p(\bigtriangledown f) = 0$ then $p(\bigtriangledown)f = 0$.* This conjecture is intimately related to a question posed by A. Pinkus and B. Wajnryb: For what homogeneous polynomials f the space $\mathcal{P}(f) := \text{span}\{f^k(\mathbf{x} + \mathbf{a}) : \mathbf{a} \in \mathbb{R}^n, k = 0, 1 \ldots\}$ is dense in $C(\mathbb{R}^n)$? In the talk I will explain the relation between two questions and verify the conjecture in the number of cases, including $n \leq 5$ and $\deg p = 2$.

On Symmetrizing the Ultraspherical Spectral Method for Self-adjoint Problems

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A mechanism is described to symmetrize the ultraspherical spectral method for self-adjoint problems. The resulting discretizations are symmetric and banded. An algorithm is presented for an adaptive spectral decomposition of self-adjoint operators. Several applications are explored to demonstrate the properties of the symmetrizer and the adaptive spectral decomposition.

Approximation of Manifolds from Scattered Data by Manifold Moving Least-squares

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In order to avoid the curse of dimensionality, frequently encountered in data analysis, there was a vast development in the field of dimension reduction techniques in recent years. Such techniques (sometimes referred to as *Manifold Learning*) assume that the scattered input data is sampled from a lower dimensional manifold, and aim at projecting it onto a lower dimensional Euclidean domain. However, this procedure inevitably introduces distortion to the geometry of the data (e.g., the curvature is lost). Furthermore, in real life applications, data is often very noisy and cannot be assumed to be situated exactly on an underlying manifold. In this work, we propose a method to approximate a d-dimensional smooth submanifold $\mathcal{M}$ residing in $\mathbb{R}^n$ ($d \ll n$) based upon scattered data points (i.e., a data cloud). We assume that the data points are sampled with noise from a lower dimensional smooth manifold and perform a non-linear Manifold Moving Least-Squares (MMLS) projection. Under some mild assumptions, the resulting approximation is shown to be infinitely smooth. And, for clean samples the approximation order is $\mathcal{O}(h^{m+1})$, where h is a local density of sample parameter (i.e., the fill distance), and m is the degree of the local polynomial approximation. In addition, our proposed implementation's complexity depends only linearly on the ambient dimension n. Therefore, this approach can be used as an alternative framework to working directly on manifold data embedded in high dimensional space, through providing access to local coordinate charts in reasonable computation time. We give some numerical examples to show the soundness of our approach in approximating manifolds as well as functions defined over manifold domains.

Degree of Approximation of the Function (Signal) with Almost Riesz Means

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A result on degree of approximation of signals belonging to the class $Lip(\alpha, p, w)$ by a almost Riesz summability method of Fourier series has been determined. This result is more applicable by using the almost Riesz summability mean in $Lip(\alpha, p, w)$. Furthermore, a set of results can be established from the main result by using the suitable conditions so some theorems become particular case of our main theorem. Degrre of approximation analysis of signals or time functions is of great importance, because it conveys information or attributes of some real life phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters in IIR and FIR filters.

Piecewise Harmonic Splines

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Bernstein-Bézier techniques for analyzing continuous harmonic splines in n variables are developed. Dimension and minimal determining sets for certain harmonic spline spaces are obtained using the new techniques. A family of conforming piecewise harmonic finite elements on the Clough-Tocher splits of triangulations is constructed as an application. The optimal order of convergence is proved for the conforming harmonic finite elements, and confirmed by numerical computations. Numerical comparisons with the standard finite elements are presented, showing advantages and disadvantages of the harmonic finite element method.

Approximate Kernel PCA: Computational vs. Statistical Trade-off

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Kernel principal component analysis (KPCA) is a popular non-linear dimensionality reduction technique, which generalizes classical linear PCA by finding functions in a reproducing kernel Hilbert space (RKHS) such that the function evaluation at a random variable X has maximum variance. Despite its popularity, kernel PCA suffers from poor scalability in big data scenarios as it involves solving an $n \times n$ eigensystem leading to a computational complexity of $O(n^3)$ with n being the number of samples. To address this issue, in this work, we consider Nyström based approximation to kernel PCA which requires solving an $m \times m$ eigenvalue problem with a computational complexity of $O(m^3 + nm^2)$, implying that the approximate method is computationally efficient if $m < n$ with m being the number of Nyström points. The goal of this work is to investigate the trade-off between computational and statistical behaviors of approximate KPCA, i.e., whether the computational gain is achieved at the cost of statistical efficiency. We show that the approximate KPCA is both computationally and statistically efficient compared to KPCA in terms of the error associated with reconstructing a kernel function based on its projection onto the corresponding eigenspaces.

Sharp Sampling Theorems in Shift-invariant Spaces

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We study the sampling problem in shift-invariant spaces $V^p(g) \subset L^p(\mathbb{R})$ generated by a univariate function $g : \mathbb{R} \to \mathbb{C}$. Two model generators are the cardinal sine function $g_1(x) = \sin(\pi x)/(\pi x)$, giving rise to the space of band-limited L^p-functions, and the cardinal B-spline $g_2 = N_m$ of polynomial degree $m-1$ with knots at the integers. On a large scale, an eminent task in Signal Processing and Approximation Theory consists in finding necessary and/or sufficient conditions for discrete sets $\Lambda \subset \mathbb{R}$, such that the sampling inequality $A_p\|f\|_p^p \le \sum_{\lambda\in\Lambda} |f(\lambda)|^p \le B_p\|f\|_p^p$ holds for all $f \in V^p(g)$, and devise algorithms for recovery of f from the samples.
For the cardinal sine g_1, an (almost) complete answer for $p = 2$ was found by A. Beurling in 1966, in terms of the so-called Beurling density. For the cardinal B-spline g_2, sufficient conditions in terms of "maximal gaps" of Λ were obtained by A. Aldroubi and K. Gröchenig in 2001, but necessary conditions

which are close to these sufficient conditions have not been found yet. For several other functions (e.g. Gaussians), the known sufficient conditions are in the form of maximal gaps. In our work we develop new methods for the consideration of a large class of smooth functions g (including Gaussians and convolutions of Gaussians with totally positive functions), such that the sampling condition can be almost characterized by the Beurling density. The method of proof uses Jensen's inequality in complex analysis and combines it with the theory of weak limits of measures as devised by Beurling.

Approximation Powers of Fourier Multiplier Operators

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Convolution operators are arguably one of the most often employed tools in function approximation theory, and play important roles in the proofs of many classical results in harmonic analysis and approximation theory, such as Jackson type inequalities, Fourier inversion formula, and Plancherel theorem, to name just a few. We study convolution operators from the broader perspective of Fourier multipliers. In particular, we use Hörmander-Mikhlin multiplier theorem to identify some easily-verifiable conditions under which a certain class of convolution operators achieve optimal approximation orders in Sobolev spaces of fractional orders.

Recovery of Linear Dynamics from Undersampled Time Series Data

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Consider a time series data $x_0, ..., x_m$ generated by linear dynamics $x_{k+1} = Ax_k$ for some unknown linear operator A. In many applications, time series data is high dimensional and full observation is very expensive. This motives us to consider recovery of A from partial observation of the time series data. This problem exhibits features that occur similar to many fundamental problems in engineering such as deconvolution, and super-resolution and the matrix completion problem. We propose reconstruction algorithms with provable guarantees by employing ideas from the classical Prony method, matrix pencil method, and the ESPRIT method.

Non-uniform Degree Splines on T-meshes: Combinatorial Bounds on the Dimension

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We provide combinatorial bounds on the dimension of non-uniform degree splines, i.e., splines with polynomial pieces of differing degrees. Such splines would allow design of complex shapes with fewer control points, i.e., cleaner and simpler designs; while for isogeometric analysis the same would lead to more efficient analysis. In particular, we study the spline space dimension on T-meshes using homological techniques introduced by Billera (1988). Doing so, we generalize the framework presented by Mourrain (2014) to the setting of both mixed polynomial degrees and mixed smoothness. When uniform polynomial degrees are chosen, the bounds coincide with those provided by Mourrain. Examples are provided to illustrate application of the theory developed.

Error Localization of Best L_1 Polynomial Approximants

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An important observation in compressed sensing is that the ℓ_0 minimizer of an underdetermined linear system is equal to the ℓ_1 minimizer when there exists a sparse solution vector. In this talk, we discuss a continuous analogue of this observation and show that the best L_0 and L_1 polynomial approximants of a polynomial that is corrupted on a set of small measure are nearly equal. We go on to demonstrate an error localization property of best L_1 polynomial approximants and use our observations to develop an improved algorithm for computing best L_1 polynomial approximants to continuous functions.

Polynomial Recovery from Mixing Data with Outliers

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Learning a generating function from a finite set of input-output observations is an important but challenging task in various scientific disciplines. In general, this data-based learning problem is ill-posed due to the nonlinearity of the unknown function and the complicated properties of given data (possibly noisy, corrupted, or correlated). One of the main directions is to investigate the sparsity-of-effect in the data-driven methods to select a suitable model. Along this direction, we study the problem of learning nonlinear functions from identically distributed but not independent data that is sparsely corrupted by outliers and noise. The learning problem is written as a parameter estimation problem where we incorporate both the unknown polynomial coefficients and the corruptions in a basis pursuit denoising framework. The main contribution of this work is to provide a reconstruction guarantee for the associated l1-optimization problem, provided that the data is bounded and satisfies a suitable concentration inequality. Applications to various type of mixing data such as chaotic data, exponentially strongly alpha-mixing data, geometrically C-mixing data, and uniformly ergodic Markov chain will also be discussed.

Null Space Conditions for Sparse Recovery via Nonconvex, Non-separable Minimizations

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For the problem of sparse recovery, it is widely accepted that nonconvex minimizations are better than ℓ_1 penalty in enhancing the sparsity of solution. However, to date, the theory verifying that nonconvex penalties outperform (or are at least as good as) ℓ_1 minimization in exact, uniform recovery has mostly been limited to separable cases. In this paper, we establish general recovery guarantees through null space conditions for nonconvex, non-separable regularizations, which are slightly less demanding than the standard null space property for ℓ_1 minimization.

Mimetic Conservation Principles for Meshfree Approximation

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While the scattered data approximation theory underpinning many meshfree discretizations provides a means of developing methods with rigorous notions of accuracy and convergence, the lack of requisite differential forms traditionally precludes the development of methods with discrete conservation principles. To illustrate, consider that meshfree methods characterize a solution purely with regard to $0-$forms on a point cloud, while conservative methods typically exploit a connection between $3-$forms and $2-$forms via the Gauss divergence theorem. We show that an $\epsilon-$ball graph of point connectivity may be endowed with metric information through the solution of inexpensive graph Laplacian problems, obtaining virtual notions of volume and face areas associated with the nodes and edges of the graph, respectively. We thus derive a mimetic divergence theorem providing a discrete conservation principle which may be used to construct traditional finite volume schemes. If time permits, we will introduce recent work extending the construction to the entire de Rham complex, illustrating how the approach generalizes to construct virtual coboundary operators in $\mathbb{R}^3$ (i.e. $div, grad, curl$) in a manner providing an exact sequence ($div \circ curl = curl \circ grad = 0$).

Probabilistic Solutions to Ordinary Differential Equations as Non-linear Bayesian Filtering

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Numerical solutions to ordinary differential equations, $\dot{y}(t) = f(y(t), t)$, can be posed as non-linear Bayesian filtering problems by defining a differentiable Gaussian process prior $X(t)$ that admits a state-space representation. For a time discretisation $\{t_n\}_{n=0}^N$, a sequence of measurements is defined by $\dot{X}(t_n) - f(X(t), t_n) = 0$. The numerical solution of an ordinary differential equation has thus been posed as a standard non-linear Bayesian filtering problem, which can be approached using standard methods from stochastic signal processing, such as Gaussian filters and particle filters. When $y \to f(y, t)$ is an affine function, the filtering problem is solved exactly by the Kalman filter and uncertainty calibration via maximum likelihood estimation is simple and closed-form. Uncertainty calibration of Gaussian filters in general can be done in a similar manner via quasi maximum likelihood estimation. Furthermore, stability on linear test equations (A-stability) for Gaussian filters can be assessed via standard results from linear filtering theory. The developed solvers are compared against other probabilistic solvers found in the literature.

Splines and Learning: From Kernel Methods to Deep Neural Nets

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Regularization is an effective strategy for dealing with the ill-posedness of the training problem in machine learning and/or the reconstruction of a signal from a limited number of measurements. The link with splines is well understood when the penalty involves a Hilbertian norm [1]. Likewise, we have recently shown that the extremal points of a broad class of linear inverse problems with generalized total-variation regularization are adaptive splines whose type is linked to the underlying regularization operator L [2]. For instance, when L is the nth derivative (resp., Laplacian) operator, the optimal reconstruction is a non-uniform polynomial (resp., polyharmonic) spline with the smallest possible number of adaptive knots. The crucial observation is that such continuous-domain solutions are intrinsically sparse, and hence compatible with the kind of formulation (and algorithms) used in compressed sensing.

We then make the link with current learning techniques by proposing to optimize the shape of individual activations in a deep neural network. By selecting the regularization functional to be the 2nd-order total variation, we obtain an optimal deep-spline network whose activations are piecewise-linear splines with a few adaptive knots [3]. Since each spline knot can be encoded with a ReLU unit, this provides a variational justification of the popular ReLU architecture. It also suggests new computational challenges for the determination of the optimal activations involving linear combinations of ReLUs.

1) C. de Boor and R. E. Lynch, "On splines and their minimum properties," *Journal of Mathematics and Mechanics*, vol. 15, pp. 953–969, 1966.

2) M. Unser, J. Fageot, J.P. Ward, "Splines Are Universal Solutions of Linear Inverse Problems with Generalized TV Regularization," *SIAM Review*, vol. 59, no. 4, pp. 769–793, December 2017.

3) M. Unser, "A Representer Theorem for Deep Neural Networks," preprint, arXiv:1802.09210, 2018.

Deep Learning Approach to Diabetic Blood Glucose Prediction

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In this talk, we consider the question of developing a deep learning algorithm for predicting a patient's blood glucose levels. More precisely, suppose that, at time $t = t_0$, we are given estimates of a patient's past blood glucose concentrations, say $s(t_{-6}), s(t_{-5}), \ldots, s(t_0)$, where $t_{-6} < \cdots < t_{-1} < t_0$, with $t_{i+1} - t_i = 5$ minutes. Our goal is to use these past measurements to predict the patient's blood glucose concentration 30 minutes in the future, at time $t = t_6$. The highlights of the talk include the design of the algorithm and numerical results that demonstrate how deep learning can outperform shallow networks in this application.

Improving Dimension Bounds for Trivariate Splines on Cells

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Finding the dimension of the space of trivariate splines on cells involves the computation of a term encoding the geometry of the faces around the central vertex. The computation of such term connects the theory of splines with the study of fat point ideals. While the dimension of fat point ideals is quite complicated, we will show that these complications happen in low enough degree and it becomes irrelevant to bounding the dimension of the spline space. Our approach is based on results by Whiteley (1991), the reduction procedure on the fat points multiplicity introduced by Cooper, Harbourne, and Teitler (2011), and the applications of these results to estimate the Waldschmidt constant for the ideal. We will present a new combinatorial lower bound on the dimension of the spline spaces on cells, and show that the contribution of the fat point ideal can be greatly simplified for most cells. We will apply the new bound to improve results by Colvin, DiMatteo, and Sorokina (2016), and illustrate our approach via examples for both generic and non-generic configurations.

Minimization of p-Frame Energies

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The p-frame energy on the unit sphere $\mathbb{S}^d$ is defined as the functional $\iint_{\mathbb{S}^d} |x \cdot y|^p \, d\mu(x) d\mu(y)$, acting on the probability measures μ supported on $\mathbb{S}^d$. The question of finding measures minimizing this energy arises naturally in signal processing: when $p = 2$, the discrete minimizers of such energies are unit norm tight frames. It is however the case of $p \notin 2\mathbb{N}$ that appears to be the most interesting. The above quadratic functional then has a kernel which is not positive definite; representatives of this class of kernels also appear in quantum-mechanical models, the problem of moments, and other areas of mathematical physics and functional analysis.
For even values of p, the uniform surface measure on $\mathbb{S}^d$ is a minimizer, and spherical designs of order p are discrete minimizers. For non-even p, the situation is more involved, in particular tight spherical designs and optimal codes arise as minimizers. We will discuss solutions for some ranges of non-even p and certain dimensions d, obtained using linear programming methods, as well as general criteria for the discreteness of minimizing measures.

Recent Advances of L^1 Splines

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Cubic splines are widely used for data interpolation and approximation in non-parametric regression, terrain surface fitting, computer aided design, numerical control, finance, healthcare and bioinformatics. Conventional splines are constructed by minimizing the L^2-norm based metrics of data fitting errors and the curvature of the spline. Such splines often show undesired oscillation and do not preserve shape, especially for irregular or multiscale data. L^1 splines, by minimization of the L1-norm based metrics, have shown superior and robust shape-preserving performances and enjoyed increasing application potentials. We introduce the development of L1 splines over the past decade, present the latest research on the fast computing strategy and the quantitative measure of shape-preserving capability, and discuss future opportunities.

A High Order Meshless Galerkin Method for Semilinear Parabolic Equations on Spheres

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This talk will describe a novel meshless Galerkin method for numerically solving semilinear parabolic equations on spheres. The new approximation method is based upon a discretization in space using spherical basis functions in a Galerkin approximation. As the spatial approximation spaces are built with spherical basis functions, they can be of arbirary order and do not require the construction of an underlying mesh. The convergence of the meshless method is established by adapting, to the sphere, a convergence result due to Thomee and Wahlbin. Numerical examples highlighting the theortical results will be presented. Work supported by DMS-1813091.

Localized Basis Functions on Graphs and Applications

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Graph domains are being used for many signal processing applications. They provide a more general framework than integer lattices, and they can be used to incorporate additional structural or geometric information. In this talk, we define intrinsic basis functions derived from the graph Laplacian, in analogy with polyharmonic splines on euclidean spaces. We consider the associated Lagrange basis functions and discuss their decay properties. The applications of such bases include kernel-based machine learning algorithms where data is well-represented using a graph framework. We shall also present some experimental results in this direction.

Learning High-dimensional Systems from Data by Nonlinear Reconstruction and Deep Learning

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This talk will focus on nonlinear approaches to sparse polynomial approximation of complex functions in high dimensions. Of particular interest is the parameterized PDE setting, where the target function is smooth, characterized by a rapidly decaying orthonormal expansion, whose most important terms are captured by a lower (or downward closed) set. By exploiting this fact, we will present and analyze several procedures for exactly reconstructing a set of (jointly) sparse vectors, from incomplete measurements, in particular:

- weighted ℓ_1 minimization procedure for compressed sensing, with a precise choice of weights, for overcoming the curse of dimensionality;
- mixed-norm based regularization that simultaneously reconstructs parameterized PDEs solutions over both physical and parametric domains;
- the first sharp estimates of the complexity of an artificial neural network required to recover the best approximation in high dimensions.

Such approaches will enable the reconstruction of the entire high-dimensional solution map, with accuracy comparable to the best approximation, while utilizing an optimal number of samples. Numerical examples are provided to support the theoretical results and demonstrate the computational efficiency of the described compressed sensing methods.

Rational Approximation in Superfast Rank-structured Solvers

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In the late 19th century, Zolotarev introduced and solved two important problems in approximation theory involving rational functions. In this talk, we show how Zolotarev's work can be used to explain a low rank matrix structure that is relied upon in several superfast Toeplitz solvers. Our results offer

a theoretically rigorous proof of the matrix structure, and as a practical benefit, they lead to new and highly efficient formulations of fully adaptive superfast solvers for Toeplitz linear systems. We also apply our ideas to develop superfast solvers for related linear systems that involve similar matrix structures.

Good Covering of the Sphere by Spherical Caps

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Consider a set of N points $\mathcal{X}_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\} \subset \mathbb{S}^d$ where $\mathbb{S}^d$ is the unit sphere in $\mathbb{R}^{d+1}$. The aim is to find sets of points which minimize the radius h for covering the sphere $\mathbb{S}^d$ by spherical caps centered at each point $\mathbf{x}_j, j = 1, \ldots, N$ and of the same radius h. This is the same as finding sets of N points which minimize the mesh norm $h_{\mathcal{X}_N} = \max_{\mathbf{x}\in\mathbb{S}^d} \min_{j=1,\ldots,N} \operatorname{dist}(\mathbf{x}, \mathbf{x}_j)$ where $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \cos^{-1}(\mathbf{x} \cdot \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathbb{S}^d$ is the geodesic distance. The mesh norm represents the largest hole in the set of points, and arises in a variety of contexts when approximating or numerically integrating functions on the sphere. This talk will concentrate on algorithms for calculating such point sets and numerical results for $\mathbb{S}^2$ and $\mathbb{S}^3$.

Localized Meshfree Semi-Lagrangian Advection Schemes for Transport on Surfaces

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Semi-Lagrangian advection (SLA) methods are particularly well-suited for advection dominated partial differential equations because they are not governed by the same stability restrictions as Eulerian based schemes (e.g. the CFL stability condition or eigenvalue stability of the discretized operators). In this talk we show how to combine the SLA framework with meshfree approximation methods for simulating the transport equation on two-dimensional surfaces embedded in $\mathbb{R}^3$. We focus on two approximation schemes based on localized radial basis function (RBF) interpolation using polyharmonic splines and polynomials. One method is formulated in the embedding space and the other in the tangent plane. Both methods are free from any artificial singularities that arise from surface-based coordinates and have $\mathcal{O}(N \log N)$ computational complexity. Additionally, the methods do not require any stabilization terms (such as hyperviscosity) during time-integration, thus reducing the number of parameters that have to be tuned. We compare the accuracy, stability, and efficiency of the two methods for several example problems on various surfaces.

Sequential Approximation Methods for Function Approximation with Big Data

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One of the central tasks in scientific computing is to accurately approximate unknown target functions. This is typically done with the help of data samples of the unknown functions. The emergence of Big Data presents both opportunities and challenges. On one hand, big data introduces more information about the unknowns and, in principle, allows us to create more accurate models. On the other hand, data storage and processing become highly challenging. In this talk, we present a set of sequential

algorithms for function approximation with extraordinarily large data sets. The algorithms are of iterative nature and involve only vector operations. They use one data sample at each step and can handle dynamic/stream data. We present both the numerical algorithms, which are easy to implement, as well as rigorous analysis for their theoretical foundation.

The Performance of the Quadratic Model for Phase Retrieval

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The quadratic model for phase retrieval is a popular model for recovering the signals from the magnitude of the measurements. One already develops many efficient algorithms for solving the model. The aim of this talk is to present the performance of the model. Particularly, we present the performance of the quadratic model for the noise suppression.

ResNet-based Isosurface Learning for Dimensionality Reduction in High-dimensional Function Approximation

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We developed a novel ResNet-based isosurface learning method for dimensionality reduction in high-dimensional function approximation. Existing methods, including sliced inverse regression, active subspace methods, ridge approximation, reduce the dimensionality by learning an affine/linear transformation; our contribution is to extend such transformation to the nonlinear regime. Specifically, we exploited the reversible ResNets to learn the target functions' isosurfaces and approximately parameterize the isosurfaces in low-dimensional spaces, so as to greatly increase the anisotropy of the original input-output map. Since the ResNet plays a different role in our method, a new loss function was designed for training the ResNets, such that a trained network can capture the nonlinearity of the isosurface. The effectiveness of our approach is demonstrated by applying it to three 2-dimensional functions for illustrating the nonlinearity of the transformation, as well as to two 20-dimensional functions for showing the improved approximation accuracy with the use of the nonlinear transformation.

Learning Physics by Data for the Motion of a Sphere Falling in a Non-Newtonian Fluid

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We will introduce a mathematical model of nonlinear jerk equation of velocity to simulate the nonuniform oscillations of the motion of a falling sphere in the non-Newtonian fluid. This differential/algebraic equation is established only by learning the experimental data with the generalized Prony method and sparse optimization method. From the numerical results, our model successfully simulates the sustaining oscillations and abrupt increase during the sedimentation of a sphere through a non-Newtonian fluid. It presents the behavior of a chaotic system which is highly sensitive to initial conditions and experimentally nonreproducible. More investigations about the normalized representation of our model show that it includes both the uniform and nonuniform oscillatory motion of the falling sphere. The

model can also be regarded as an elastic system comprising of the flow-induced structure formed in the shear region around the sphere and the extensional stress in the wake of the sphere. It is consistent with the evidence from physical experiments.

Non-polynomial Divided Differences and Generalized Taylor Series

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The divided differences are commonly used in numerical analysis and approximation theory, and are related to both Newton interpolation and B-spline approximation. They are directly involved in the definition of B-splines. Recently, Zürnacı and Dişibüyük give an explicit representation of non-polynomial B-spline functions for a wide collection of spline spaces including trigonometric splines, hyperbolic splines, and special Müntz spaces of splines by using non-polynomial divided differences applied to a proper generalization of truncated-power function. Some properties of non-polynomial divided differences such as symmetry and Leibniz formula are obtained. With the definition of a generalized derivative operator, it is shown that as in the polynomial case, non-polynomial divided differences can be viewed as a discrete analogue of derivatives. In this study, we obtain a generalization of Taylor series using non-polynomial divided difference. Also, it is shown that non-polynomial divided differences possess some identities related to cancellation, affine combinations and ratio of determinants.