Planar graphs with girth at least 5 are $(3, 4)$-colorable

Xia Zhang
pandarhz@sina.com

Shandong Normal University, China

Coauthors: Ilkyoo Choi and Gexin Yu

May 20, 2017

Introduction

- A proper r-coloring of graph is a coloring of the graph with r colors so that each color class forms an independent set.

Introduction

- A proper r-coloring of graph is a coloring of the graph with r colors so that each color class forms an independent set.

- On proper coloring of planar graphs, a famous example is the Four Color Theorem.

Introduction

- A proper r-coloring of graph is a coloring of the graph with r colors so that each color class forms an independent set.
- On proper coloring of planar graphs, a famous example is the Four Color Theorem.
- We may relax the requirement by allowing some edges in each color class.

Introduction–1

- A graph G is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i \in \{1, \ldots, r\}$, where d_is are non-negative integers.

Introduction–1

- A graph G is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i \in \{1, \ldots, r\}$, where d_is are non-negative integers.

- Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. (See Montassier and Ochem, Near-colorings: non-colorable graphs and NP-completeness, the electronic journal of combinatorics 22(1) (2015), $\#P1.57$)

Introduction–1

- A graph G is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i \in \{1, \ldots, r\}$, where d_is are non-negative integers.

Introduction–1

- A graph G is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i \in \{1, \ldots, r\}$, where d_is are non-negative integers.

- The Four Color Theorem says that every planar graph is $(0, 0, 0, 0)$-colorable.

Introduction–1

- A graph G is called $(d_1, d_2, \ldots, d_r)$-colorable, if its vertex set can be partitioned into r nonempty subsets so that the subgraph induced by the ith part has maximum degree at most d_i for each $i \in \{1, \ldots, r\}$, where d_is are non-negative integers.
- The Four Color Theorem says that every planar graph is $(0, 0, 0, 0)$-colorable.
- In 1986, Cowen, Cowen, and Woodall proved that planar graphs are $(2, 2, 2)$-colorable. In 1999, Eaton and Hull, Škrekovski, separately, proved that this is sharp by exhibiting non-$(1, k, k)$-colorable planar graphs for each k. Thus, the problem is completely solved when $r \geq 3$.

Introduction–2

- The girth of a graph G is the length of a shortest cycle. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g.

Introduction–2

- The girth of a graph G is the length of a shortest cycle. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g.
- There are non-(d_1, d_2)-colorable planar graphs in $\mathcal{G}_4$ for any d_1, d_2. (Montassier and Ochem, 2015)

Introduction–2

- The girth of a graph G is the length of a shortest cycle. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g.
- There are non-(d_1, d_2)-colorable planar graphs in $\mathcal{G}_4$ for any d_1, d_2. (Montassier and Ochem, 2015)
- There are non-$(0, k)$-colorable planar graphs in $\mathcal{G}_6$ for any k. (Borodin, Ivanova, Montassier, Ochem and Raspaud, 2010)

Introduction–2

- The girth of a graph G is the length of a shortest cycle. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g.
- There are non-(d_1, d_2)-colorable planar graphs in $\mathcal{G}_4$ for any d_1, d_2. (Montassier and Ochem, 2015)
- There are non-$(0, k)$-colorable planar graphs in $\mathcal{G}_6$ for any k. (Borodin, Ivanova, Montassier, Ochem and Raspaud, 2010)
- There are non-(2, 0)-colorable planar graphs in $\mathcal{G}_7$. (Montassier and Ochem, 2015)

Introduction–2

- The girth of a graph G is the length of a shortest cycle. Let $\mathcal{G}_g$ denote the class of planar graphs with girth at least g.
- There are non-(d_1, d_2)-colorable planar graphs in $\mathcal{G}_4$ for any d_1, d_2. (Montassier and Ochem, 2015)
- There are non-$(0, k)$-colorable planar graphs in $\mathcal{G}_6$ for any k. (Borodin, Ivanova, Montassier, Ochem and Raspaud, 2010)
- There are non-(2, 0)-colorable planar graphs in $\mathcal{G}_7$. (Montassier and Ochem, 2015)
- There are non-$(3, 1)$-colorable planar graphs in $\mathcal{G}_5$. (Montassier and Ochem, 2015)

Some known results on (d_1, d_2)-colorable graphs in $\mathcal{G}_5$

- Planar graphs in $\mathcal{G}_5$ are $(1, 10)$-colorable. (Choi, Choi, Jeong and Suh 2016)

Some known results on (d_1, d_2)-colorable graphs in $\mathcal{G}_5$

- Planar graphs in $\mathcal{G}_5$ are $(1, 10)$-colorable. (Choi, Choi, Jeong and Suh 2016)
- Planar graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable. (Borodin and Kostochka 2014)

Some known results on (d_1, d_2)-colorable graphs in $\mathcal{G}_5$

- Planar graphs in $\mathcal{G}_5$ are $(1, 10)$-colorable. (Choi, Choi, Jeong and Suh 2016)
- Planar graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable. (Borodin and Kostochka 2014)
- Planar graphs in $\mathcal{G}_5$ are $(3, 5)$-colorable. (Choi and Raspaud 2015)

Some known results on (d_1, d_2)-colorable graphs in $\mathcal{G}_5$

- Planar graphs in $\mathcal{G}_5$ are $(1, 10)$-colorable. (Choi, Choi, Jeong and Suh 2016)
- Planar graphs in $\mathcal{G}_5$ are $(2, 6)$-colorable. (Borodin and Kostochka 2014)
- Planar graphs in $\mathcal{G}_5$ are $(3, 5)$-colorable. (Choi and Raspaud 2015)
- Planar graphs in $\mathcal{G}_5$ are $(4, 4)$-colorable. (Havet and Sereni 2006)

A summary on (d_1, d_2)-coloring

girth	$(k,0)$	$(k,1)$	$(k,2)$	$(k,3)$	$(k,4)$
$3,4$	X	X	X	X	X
5	X	$(10,1)$	$(6,2)$	$(5,3)$	$(4,4)$
6	X	$(4,1)$	$(2,2)$		
7	$(4,0)$	$(1,1)$			
8	$(2,0)$				
11	$(1,0)$				

A summary on (d_1, d_2)-coloring

girth	$(k, 0)$	$(k, 1)$	$(k, 2)$	$(k, 3)$	$(k, 4)$
$3, 4$	X	X	X	X	X
5	X	$(10, 1)$	$(6, 2)$	$(4, 3)$	$(3, 4)$
6	X	$(4, 1)$	$(2, 2)$		
7	$(4, 0)$	$(1, 1)$			
8	$(2, 0)$				
11	$(1, 0)$				

Our result and its proof

- **Theorem 1.(Choi, Yu and Z., 2017^{+})** Planar graphs with girth at least 5 are $(3,4)$-colorable.

Our result and its proof

- **Theorem 1.(Choi, Yu and Z., 2017^{+})** Planar graphs with girth at least 5 are $(3,4)$-colorable.

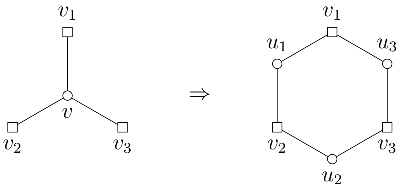
- **Proof.** Let G be a counterexample to Theorem 1 with the minimum number of 3^{+}-vertices, and subject to that choose one with the minimum number of edges.

Our result and its proof

- **Theorem 1.(Choi, Yu and Z., 2017$^+$)** Planar graphs with girth at least 5 are $(3, 4)$-colorable.
- **Proof.** Let G be a counterexample to Theorem 1 with the minimum number of 3^+-vertices, and subject to that choose one with the minimum number of edges.
- **Claim.** G must be connected and there are no 1-vertices in G.

Our result and its proof

- **Theorem 1.(Choi, Yu and Z., 2017$^+$)** Planar graphs with girth at least 5 are $(3,4)$-colorable.
- **Proof.** Let G be a counterexample to Theorem 1 with the minimum number of 3^+-vertices, and subject to that choose one with the minimum number of edges.
- **Claim.** G must be connected and there are no 1-vertices in G.
- **Lemma 2** There is no 3-vertex in G.



Proof–1

- Let the initial charge of each element $x \in V \cup F$ be $\mu(x) = d(x) - 4$. Then by Euler formula,

$$\sum_{x \in V \cup F} \mu(x) = -8.$$

Proof–1

- Let the initial charge of each element $x \in V \cup F$ be $\mu(x) = d(x) - 4$. Then by Euler formula,

$$\sum_{x \in V \cup F} \mu(x) = -8.$$

- By design discharging rules, we will show that the sum of the charges is non-negative after the discharging is finished.

Proof–1

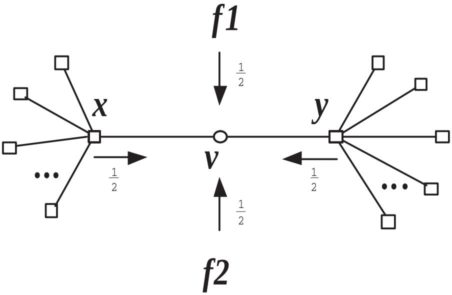
- Let the initial charge of each element $x \in V \cup F$ be $\mu(x) = d(x) - 4$. Then by Euler formula,

$$\sum_{x \in V \cup F} \mu(x) = -8.$$

- By design discharging rules, we will show that the sum of the charges is non-negative after the discharging is finished.

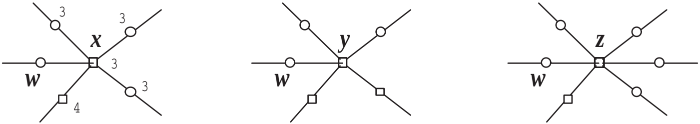
- Clearly, by Lemma 2, each face and each vertex has a non-negative initial charge except 2-vertices.

Proof–3



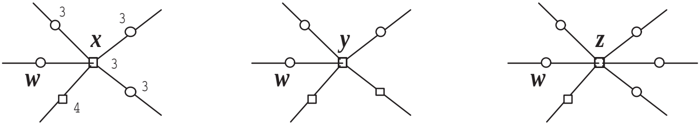
Proof–4

- Three special vertices:
  5p-vertex x, 5s-vertex y and 6p-vertex z.



Proof–4

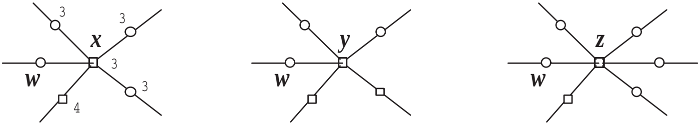
- Three special vertices:
  5p-vertex x, 5s-vertex y and 6p-vertex z.



- $\mu'(x) = 5 - 4 - \frac{4}{2} = -1.$

Proof–4

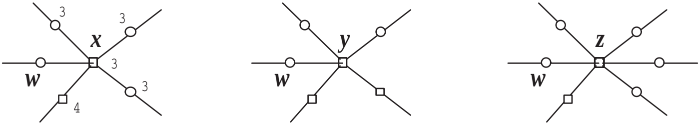
- Three special vertices:
  5*p*-vertex x, 5*s*-vertex y and 6*p*-vertex z.



- $\mu'(x) = 5 - 4 - \frac{4}{2} = -1.$
- $\mu'(y) = 5 - 4 - \frac{3}{2} = -\frac{1}{2}.$

Proof–4

- Three special vertices:
  5p-vertex x, 5s-vertex y and 6p-vertex z.



- $\mu'(x) = 5 - 4 - \frac{4}{2} = -1.$
- $\mu'(y) = 5 - 4 - \frac{3}{2} = -\frac{1}{2}.$
- $\mu'(z) = 6 - 4 - \frac{5}{2} = -\frac{1}{2}.$

Proof+

- So $5p$-, $5s$- and $6p$-vertices need get extra charge from their neighbors with high degree and incident faces with remaining charge.

Proof+

- So $5p$-, $5s$- and $6p$-vertices need get extra charge from their neighbors with high degree and incident faces with remaining charge.
- Considering these three special vertices and some special faces, we design the discharging rules.

Proof+

- So $5p$-, $5s$- and $6p$-vertices need get extra charge from their neighbors with high degree and incident faces with remaining charge.
- Considering these three special vertices and some special faces, we design the discharging rules.
- By the discharging rules, there is $\mu^*(x) \geq 0$ for each $x \in V \cup F$. So we have

$$\sum_{x \in V \cup F} \mu^*(x) \geq 0,$$

a contradiction.

Some problems

- **Problem 1.** Given a pair (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

Some problems

- **Problem 1.** Given a pair (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.
- **Problem 2.** Given a pair (g, d_1), determine the minimum $d_2 = d_2(g, d_1)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

Some problems

- **Problem 1.** Given a pair (d_1, d_2), determine the minimum $g = g(d_1, d_2)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

- **Problem 2.** Given a pair (g, d_1), determine the minimum $d_2 = d_2(g, d_1)$ such that every planar graph with girth g is (d_1, d_2)-colorable.

- **Problem 3.** What is the minimum d where graphs with girth 5 are $(3, d)$-colorable in $\{2, 3, 4\}$?

Thank you for your attention!