A notion of minor-based matroid connectivity

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Matroids in the language of graph theory

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Definition (Tutte)

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A graph is (Tutte) *k-connected* if there is no k'-separation with $k' < k$.

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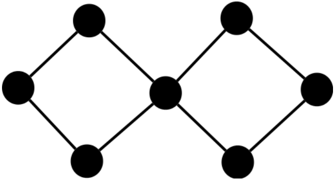
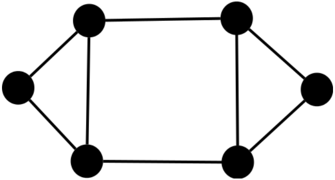


not connected



connected but not 3-connected

Connectivity

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Equivalently, a matroid M is connected if, for every pair of elements e, f of $E(M)$, there is a $M(C_2)$-minor using $\{e, f\}$.

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Example: If $N = M(C_3)$:

- If M is N-connected, it must be connected.
- If M is N-connected, it must be simple.
- If M is 2-connected and simple, then every pair of elements is some cycle of size at least 3. Therefore they are in an $M(C_3)$-minor together, so TONCAS.

Another result of Tutte

Theorem
If M is connected, then for every e of $E(M)$, one of $M\backslash e$ or M/e is also connected.

Uniform matroids

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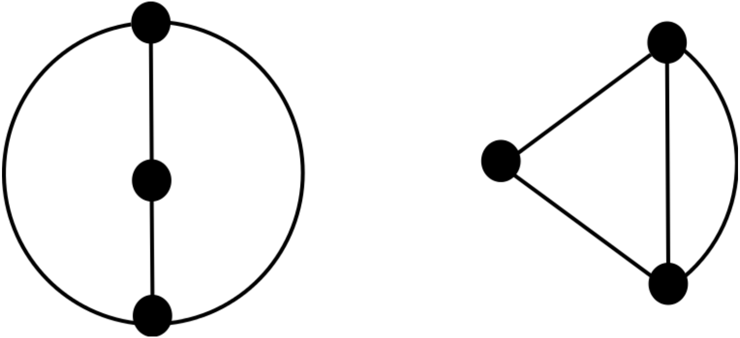
Examples:

- An n-element cycle C_n gives the matroid $U_{n-1,n}$.
- Its dual graph gives the matroid dual $U_{1,n}$.

$M(\mathcal{W}_2)$



Two drawings of $\mathcal{W}_2$

$M(\mathcal{W}_2)$-connectivity

Theorem (G., Oxley 2017)

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If a matroid is connected and non-uniform, proof by induction. Try to get $\{x, y\} \subset E(M)$ into an $M(\mathcal{W}_2)$. If there is an $e \notin \{x, y\}$ such that M/e is disconnected, M is a parallel connection of two matroids along e.

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So suppose there is no e such that $M\backslash e$ or M/e is disconnected. If $M\backslash e$ is uniform, that means e is in a non-spanning cycle, so M/e is non-uniform.

Transitivity lemma

Lemma

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Example: If M is N-connected, and N is connected and simple (that is, $M(C_3)$-connected), then M is connected and simple.

N-connected minors

Theorem

Any N-connected matroid M will have that one of its minors $M\backslash e$ or M/e is also N-connected if and only if $N \in \{U_{1,2}, U_{0,2}, U_{2,2}\}$.

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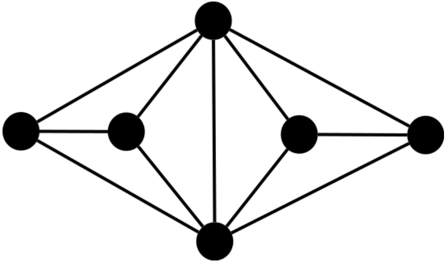
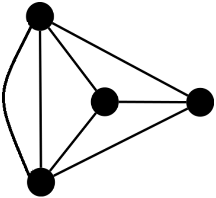
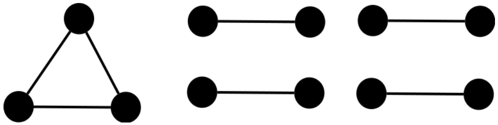
Proof sketch. Suppose N is connected. Glue together copies of N and take minors to show that N cannot be simple, cosimple, or non-uniform.





N-connected minors (with disconnected N)

Now suppose N is disconnected. If N is just loops and coloops, consider:



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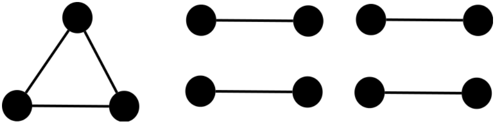

If N has a component with size ≥ 2, let N' be the parallel connection of all such components, and let N'' be another copy of N. Then $M = N' \oplus N''$ is N-connected, but if we remove all elements form N' except one, then remove an element from the largest component of N'', the resulting matroid has size $|N|$ but it has too many 1-element components.

$U_{0,1} \oplus U_{1,1}$-connectivity.

$U_{0,1} \oplus U_{1,1}$ is the matroid of $C_1 \oplus K_2$.

Theorem

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Elements are clones if interchanging them gives **the same** (not just isomorphic!) matroid. This is true if and only if they are in precisely the same set of dependent flats.

Suppose e is in a dependent flat $F \subseteq E(M)$ and f is not. Contract $F - e$. Then e will be a loop and f will not be. Delete the remaining matroid except for $\{e, f\}$ to obtain $U_{0,1} \oplus U_{1,1}$.

Other results

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Equivalently, if e is in a $M(C_2)$-minor with f ...

The only matroids with this property are $U_{1,2}$ and $M(\mathcal{W}_2)$ (connected and non-uniform).

Other results (continued)

Theorem

Let N be a 3-connected matroid. Then M is N-connected if and only if, in the Cunningham-Edmonds tree decomposition T of M, every vertex of T that is not N-connected has at most one element of $E(M)$, and if v and u are vertices of T having exactly one element of $E(M)$, the path between v and u in T has an N-connected vertex.

Summary

- *N-connectivity* is defined as when every pair of elements is in an N-minor.
- *$U_{2,3}$-connectivity* means connected and simple.
- *$M(\mathcal{W})_2$-connectivity* means connected and non-uniform.
- *$U_{1,2}$-connectivity* is normal matroid (2)-connectivity, which is unique for a number of reasons.
- *$U_{0,1} \oplus U_{1,1}$-connectivity* means no clones.

References

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Thank you!