Obstacle Numbers of Some Ptolemaic Graphs

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Outline

1. Obstacles
2. Ptolemaic Graphs
3. Results So Far
4. The Difficulties
5. Open Problems

Obstacles

Definition (Obstacle Representation of a Graph)

Consider a graph whose vertices are points in the plane along with a set of polygonal obstacles. Two vertices are adjacent if the straight line connecting the points in the plane do not intersect an obstacle.
An *obstacle representation* of a graph is the set of points and polygons.

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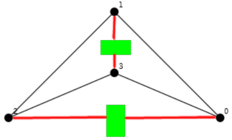
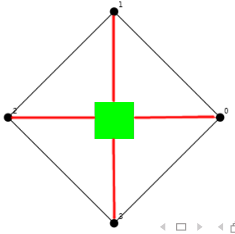
Note that an obstacle representation is not necessarily unique.





Obstacle Number

Definition (Obstacle Number of a Graph)

The *obstacle number* of a graph G, denoted obs(G) is the minimum number of obstacles such that an obstacle representation of the graph exists.

There are some classes of graphs with trivial-to-compute obstacle numbers.

- The complete graphs K_n are the only graphs with obstacle number 0.
- Complete graphs minus an edge have obstacle number 1.
- Trees have obstacle number 1.
- Cycles have obstacle number 1.

Known Results for Obstacle Numbers

Theorem (Chaplick, Lipp, Park, Wolff, 2016)

All graphs on 7 or fewer vertices are either the complete graph or have obstacle number 1.

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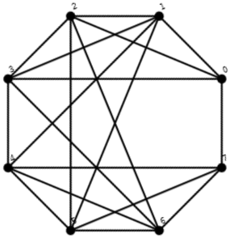

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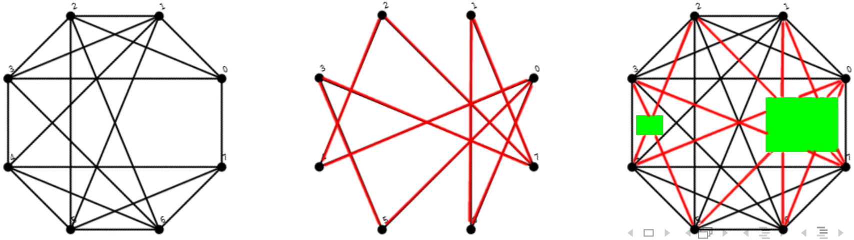

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Theorem (Berman, Chappell, Faudree, Gimbel, Hartman, Williams, 2016)

If a graph is not the complete graph, then adding a pendant vertex (vertex of degree 1) does not increase the obstacle number. If the graph is complete, then adding a pendant vertex increases the obstacle number by 1.

This last result is what started us thinking about Ptolemaic graphs.

Ptolemaic Graphs

Definition (True Twin)

A vertex, v' is a *true twin* to a vertex v if $N(v') = N(v)$ and $v'v \in E(G)$.

Definition (False Twin)

A vertex, v' is a *true twin* to a vertex v if $N(v') = N(v)$ and $v'v \notin E(G)$.

Definition (Ptolemaic Graph)

A *Ptolemaic graph* is a graph that can be constructed from a single vertex by repeated use of three operations:

1. Adding a pendant vertex to a vertex.
2. Adding a *true twin* to a vertex.
3. Adding a *false twin* to a vertex whose neighborhood a clique.

Transformations

Obstacle Preserving Transformations

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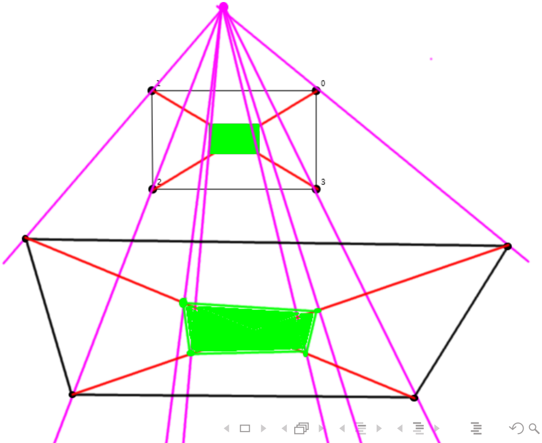

Ptolemaic* Graphs

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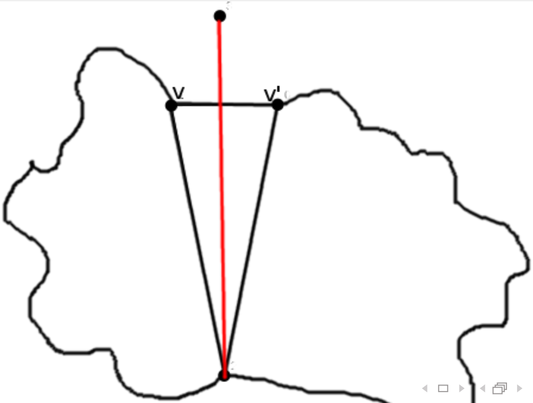
We denote this class of graphs as P_t^* .

Results So Far

Lemma (B, Dean, 2017+)

Adding a true twin vertex does not increase the obstacle number.



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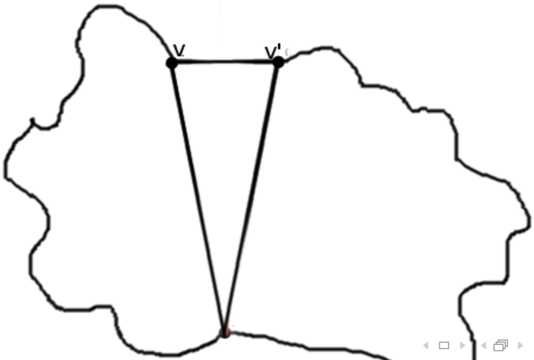

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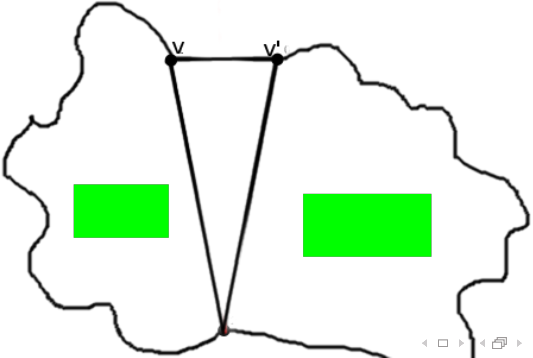


The Main Result

Theorem (B, Dean, 2017+)

If a Ptolemaic graph is not the complete graph, then it has obstacle number 1.*

Sketch of the proof.

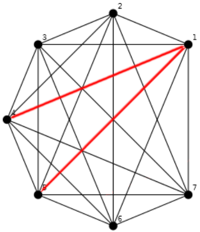
- Induct on the number of vertices. The base case is that all graphs on 7 or fewer vertices are complete or have obstacle number 1.
- Look at a Ptolemaic* graph on $n+1$ vertices (not K_{n+1}).
  - If it has a pendant vertex, remove it. Berman et al says we can put it back.
  - If there is no pendant vertex, it must have a true twin which can be removed. Our lemma says we can put it back.

□

The complete graph case is even easier.

False Twins

What about False Twins?



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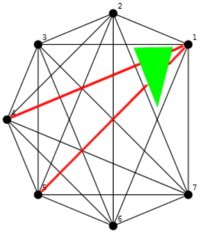

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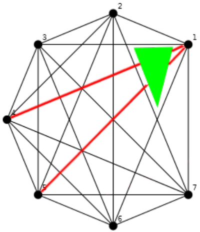

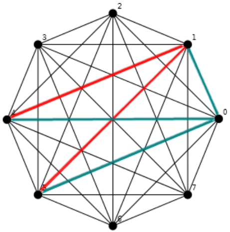

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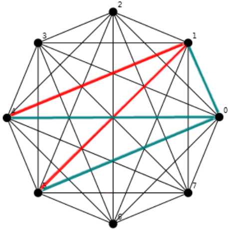

$obs(P_t^*) \leq 1$
Brauch, Dean
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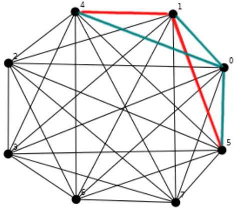

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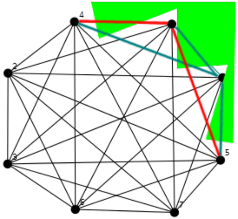

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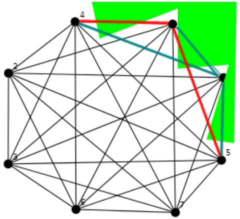

Conjecture

If the neighborhood of a vertex v is a <u>clockwise consecutive</u> clique, then you can add a false twin to v.

Other Open Problems

- Are all Ptolemaic graphs obstacle 1 graphs?
- Are all distance hereditary graphs obstacle 1 graphs?
- Trees and Complete graphs are extremes. How many edges allow for an graph with obstacle number 2?

Questions?

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