Graph Minors: When Being Shallow is Hard

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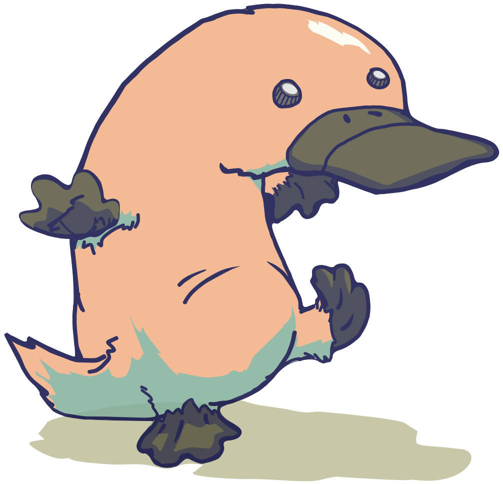
Motivation: Excluded Substructures

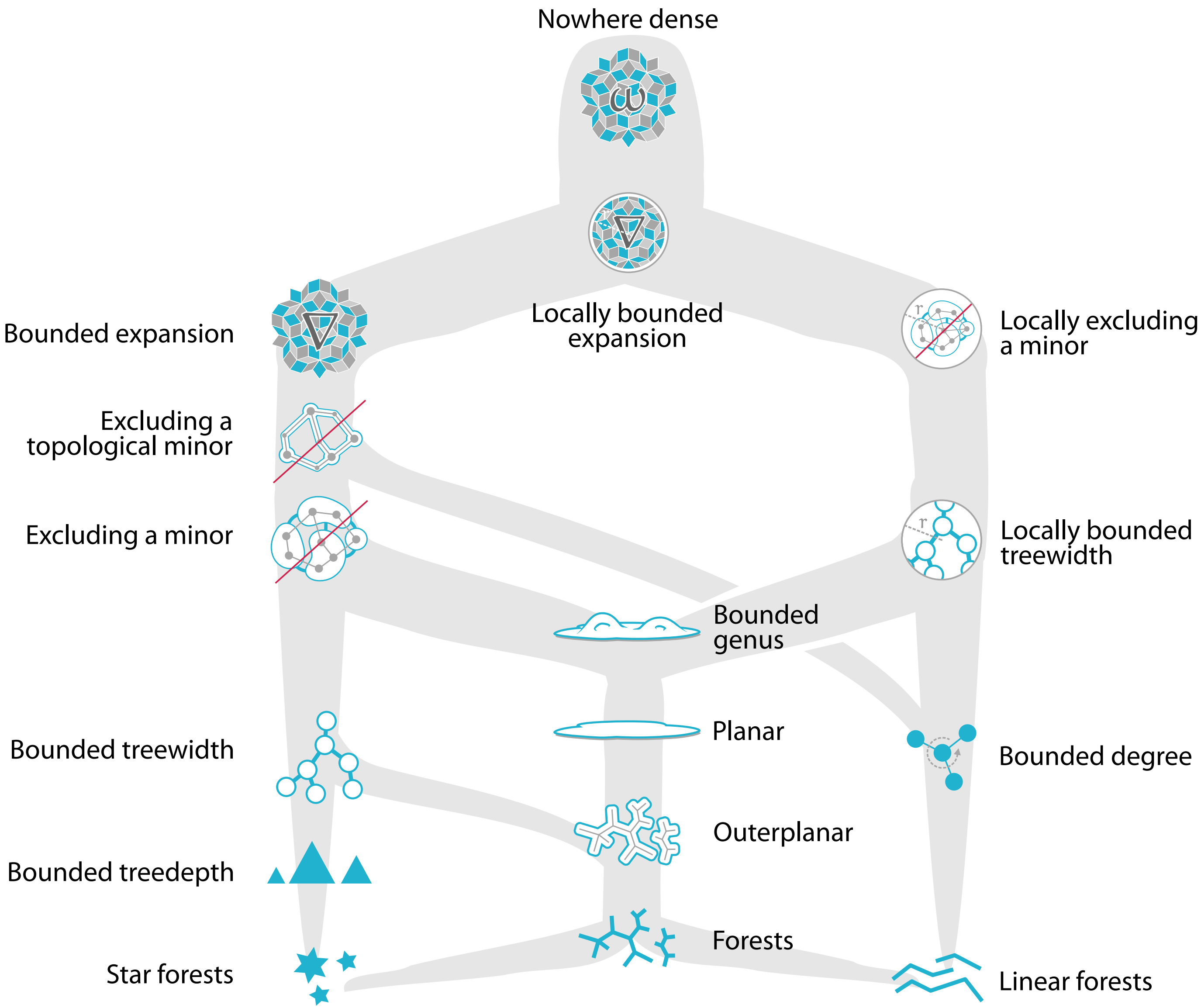


- Structural Graph Theory:
  - Forbidden Graph Characterizations
  - Turan-type Problems
  - Erdos-Hajnal Conjecture

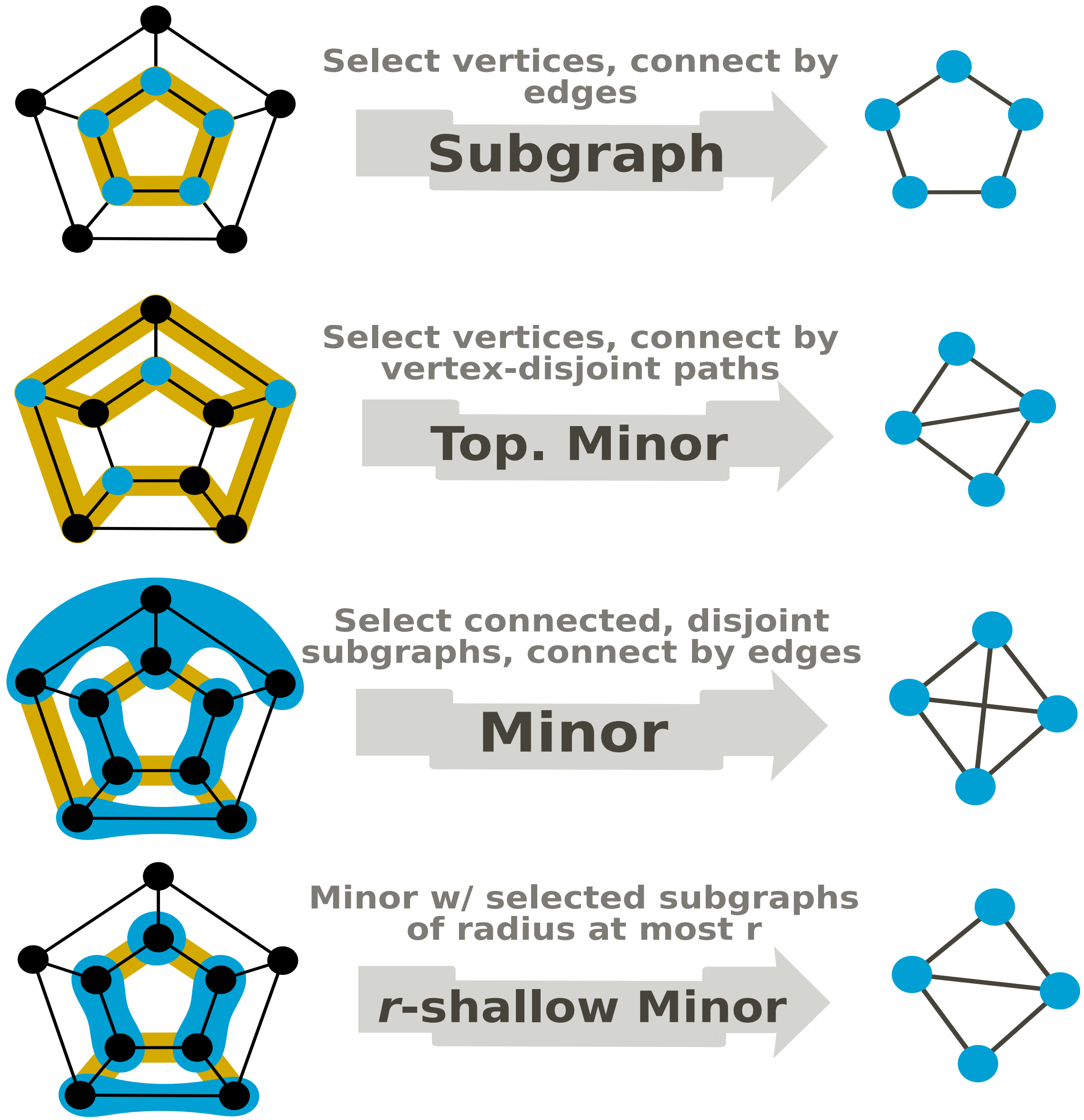
Algorithmic consequences!

- Robertson & Seymour: Graph Minors
  - Parameterized Complexity
  - Bidimensionality
  - Meta-Theorems (FPT algorithms for FO-/MSO-logics)
- Nešetřil & Ossona de Mendez: Sparse Classes
  - Bounded Expansion, Nowhere Dense

Sparse Graphs: Dense Substructures



A few definitions



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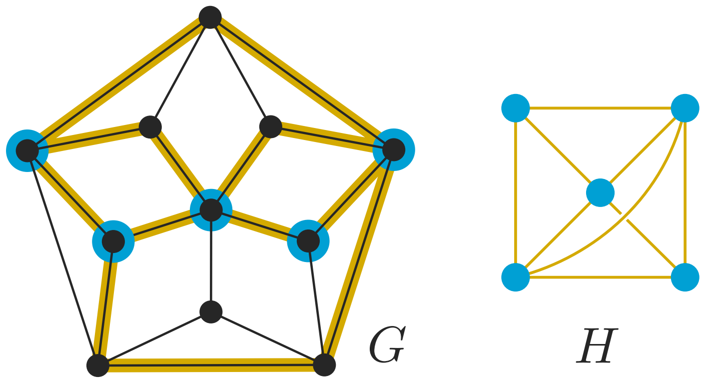
- **Subgraphs:** Surprise! This is efficiently computable with flow-based methods (Gallo et al, Goldberg).

Shallow Topological Minors & Subdivisions

- *r*/2-shallow top. minors (STM): paths of length at most *r*
- *r*-subdivision (SD): paths of length exactly *r*

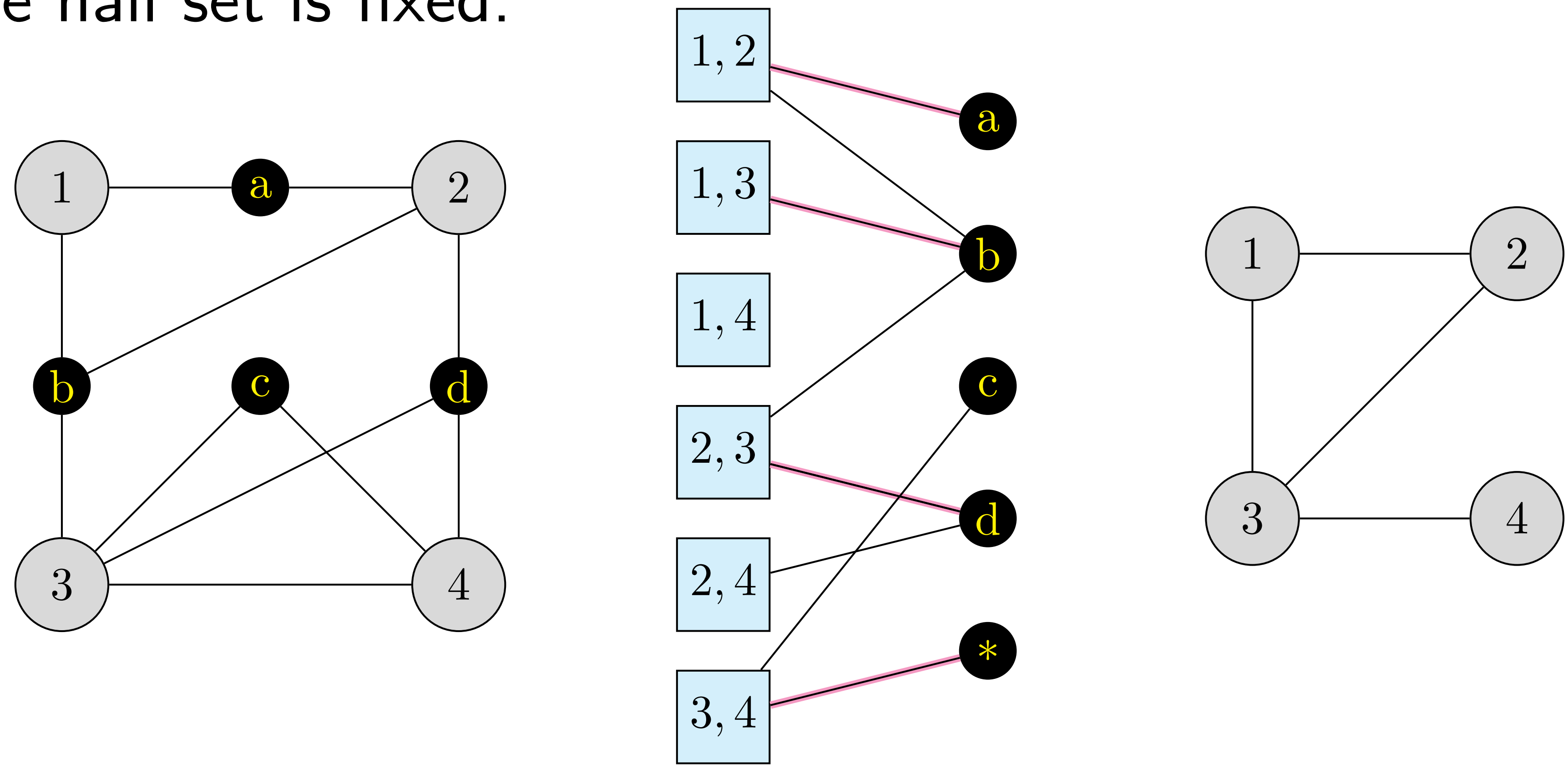
Models consist of
subdivision vertices & nails



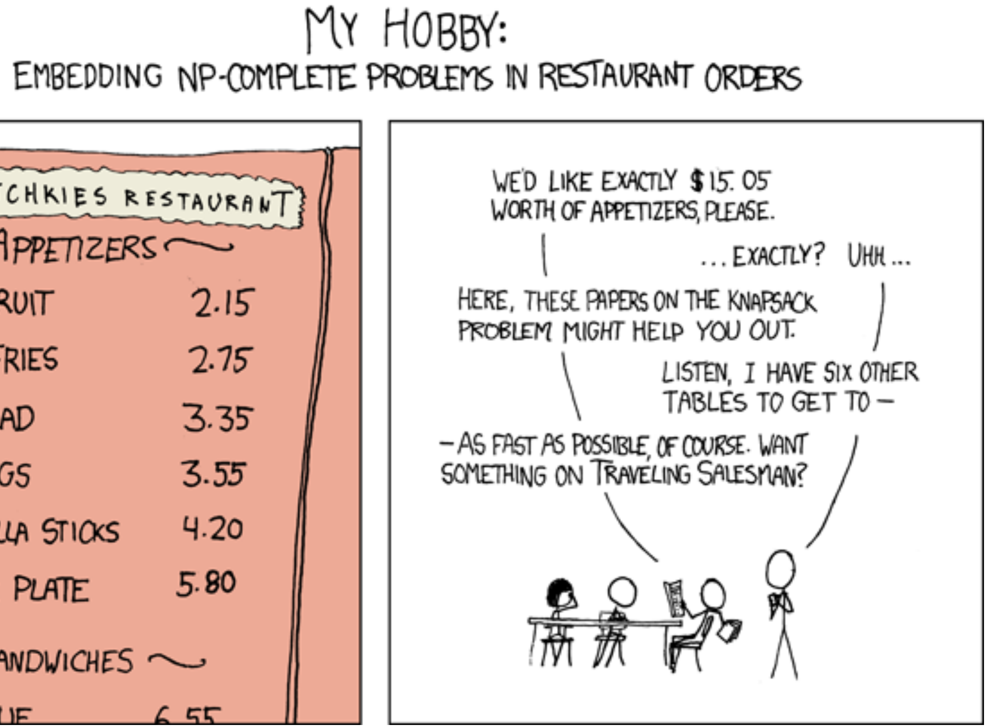
½-shallow and 1-shallow top. minors are more general than subgraphs, but more local than 1-shallow minors – *can we find dense ones in poly-time*?

If I had a hammer (when you know the nails)

Theorem: There is an $O^*(2^n)$ algorithm for DENSEST-½-SHALLOWTOPMINOR (and 1-SD) when the nail set is fixed.



It's never as easy as it seems

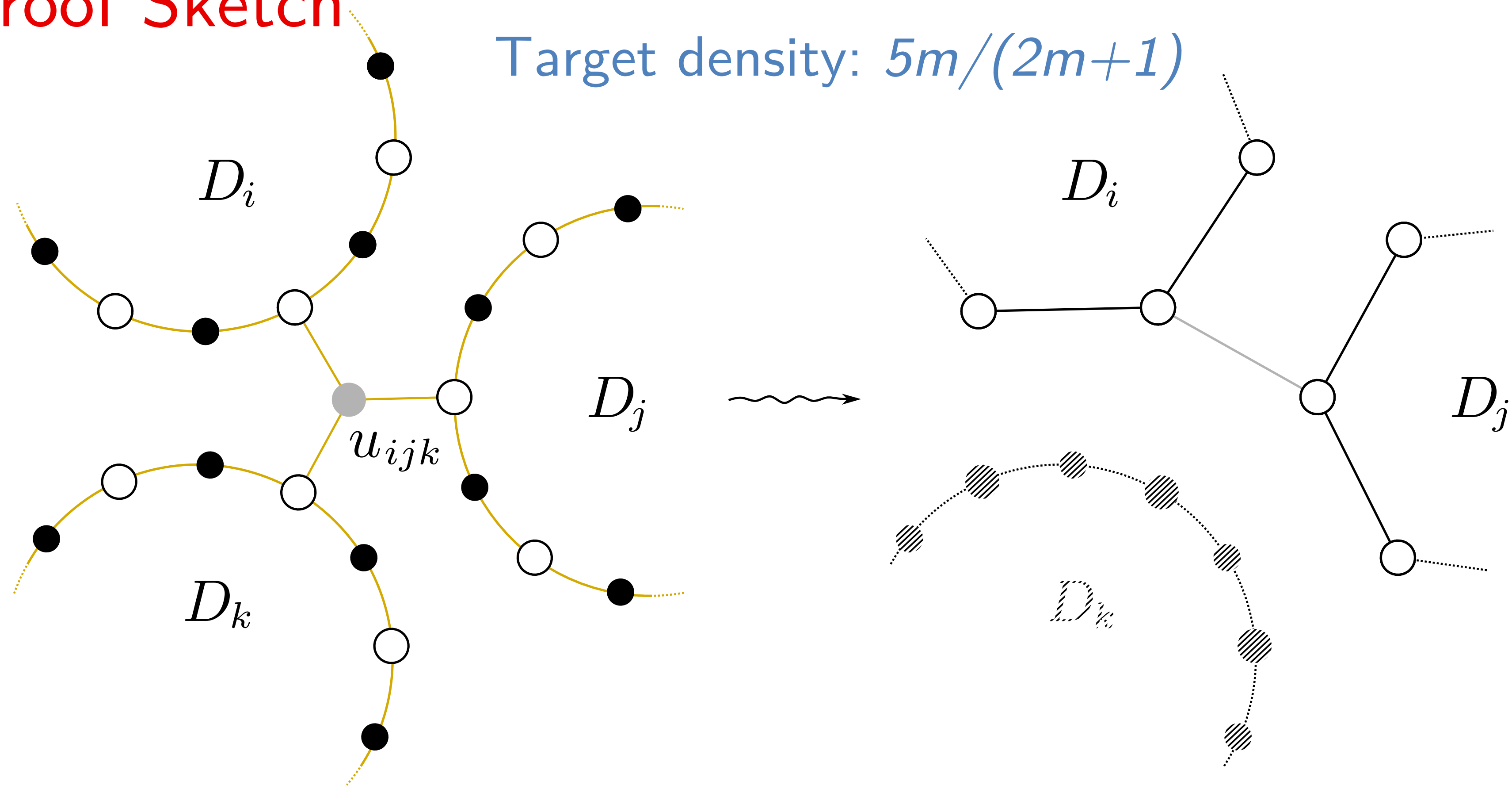


Theorem: DENSE-$r/2$-SM and DENSE-r-SD are NP-hard for $r \geq 1$, even on subcubic planar graphs plus an apex.

Idea: reduce from POSITIVE 1-IN-3SAT (which has a linear reduction from 3SAT and is NP-hard even on planar formulas). So now we get to gadgeteer!

Proof Sketch

Target density: *5m/(2m+1)*



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- *Variables become cycles with subdivided edges*
- *"Apex" attaches to cycle vertices*

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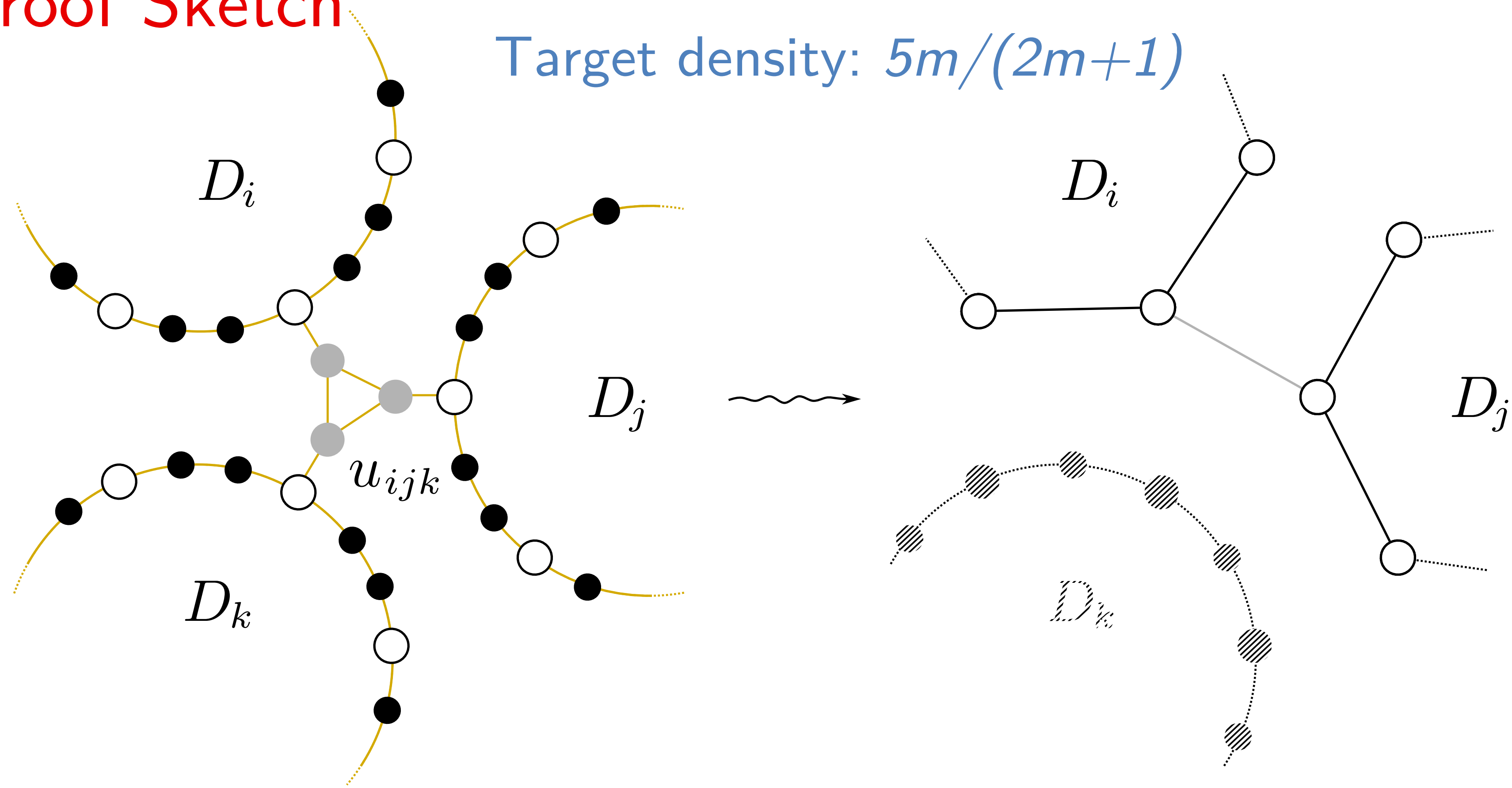
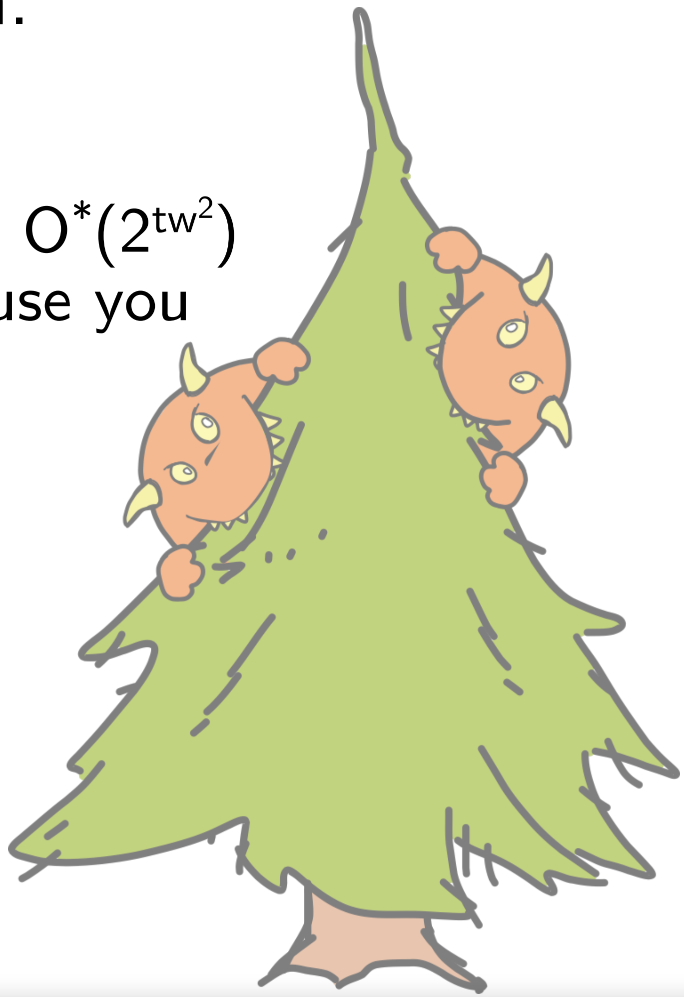


- *Clauses become claws – with center vertex replaced by triangle*
- *Variables become cycles with subdivided edges*
- *"Apex" attaches to cycle vertices*

What if the treewidth is bounded?

Theorem: DENSE-$r/2$-STM and DENSE-r-SD are FPT parameterized by treewidth.

It's tedious (but not "hard") to describe a $O^*(2^{tw^2})$ algorithm – quadratic dependence is because you have to keep track of which edges you've contracted.



Theorem: DENSE-1-STM has no $2^{o(tw^2)} n^{O(1)}$ algorithm (unless ETH fails).

ETH lower bounds

"There are no subexponential algorithms for 3SAT*"*

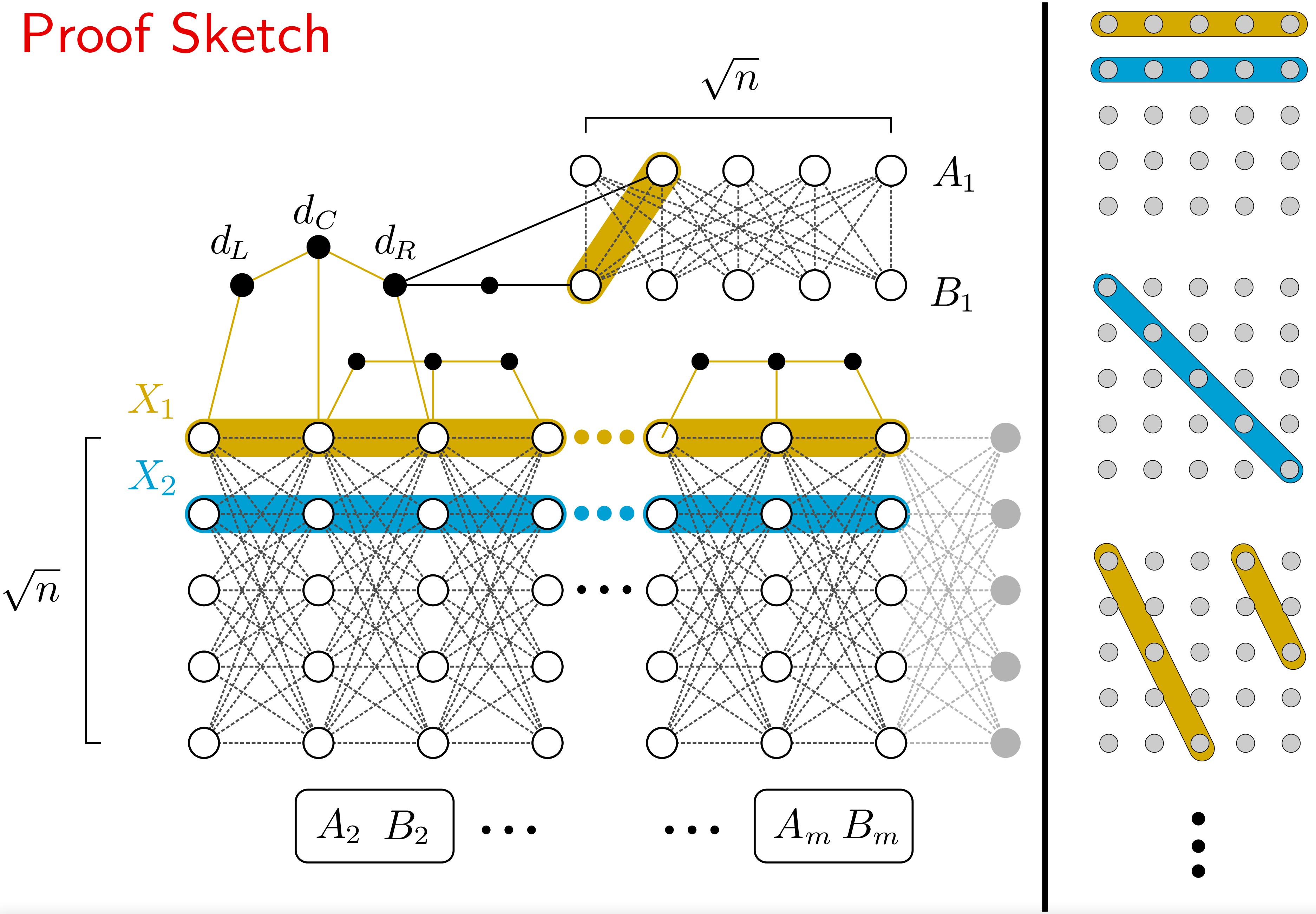
Exponential Time Hypothesis [Impagliazzo et al, 1999]

There is a positive real s such that 3SAT with n variables and m clauses cannot be solved in time $2^{sn}(n+m)^{O(1)}$.

This enables lower bounds on the complexity of problems in graphs of bounded treewidth:

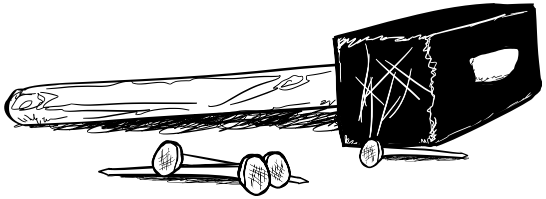
1) Do a standard NP-hardness reduction from 3SAT
2) Show the graph has treewidth $O(\sqrt{n})$
3) Now, if you could do DP to solve the problem in $O(2^{tw})$, we could run it on the reduction graph and solve SAT in $O(2^{\sqrt{n}})$, contradicting ETH

Proof Sketch



Open Questions

- Can you beat our $O^*(2^n)$ algorithm for ½-STM (e.g. $O^*((2\text{-}\varepsilon)^n)$)? If not, can you prove a SETH lower bound?
- Is ½-STM easier than 1-STM in bounded treewidth? Or is there an ETH lower bound on ½-STM showing $O^*(2^{tw^2})$ is best possible?
- Is there a (sensible) structure between ½-STM and subgraphs where we can find the densest occurrence in poly-time?



This work is under review; the preprint is available on the ArXiv: arvix.org/abs/1705.06796, "Being even slightly shallow makes life hard"

Shameless Plug

We're Hiring!

Postdoc positions available!

2-4 openings likely in 2017-2020.

Know a great undergrad?

Encourage them to apply to NC State CSC and list me as faculty they're interested in working with!