Strong chromatic index of graphs with maximum degree four

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Joint Work with M. Huang and G. Yu

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Given a graph G, a **strong edge-coloring** is a coloring of $E(G)$ such that every color class forms an induced matching in G.

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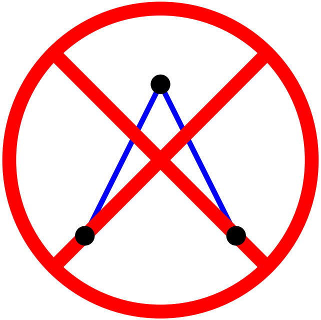

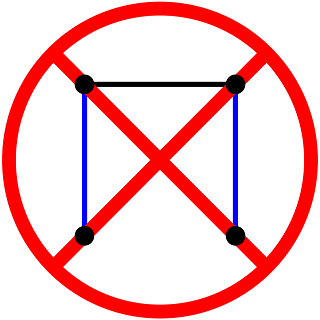

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The **strong chromatic index** of G, denoted by $\boldsymbol{\chi'_s(G)}$, is the minimum number of colors needed for a strong edge-coloring of G.

Bounds

Proposition

For every graph G with maximum degree Δ,

$$\Delta \leq \chi'(G) \leq \chi'_s(G)$$

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For every graph G with maximum degree Δ,

$$\Delta \leq \chi'(G) \leq \chi'_s(G) \leq 2\Delta(\Delta - 1) + 1.$$

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$$\Delta \leq \chi'_s(G) \leq 2\Delta^2.$$

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- The lower bound is best possible due to $K_{1,\Delta}$.
- The order of magnitude of the upper bound is also best possible as

$$\chi'_s(K_{\Delta+1}) = \binom{\Delta+1}{2} \approx \frac{1}{2}\Delta^2.$$

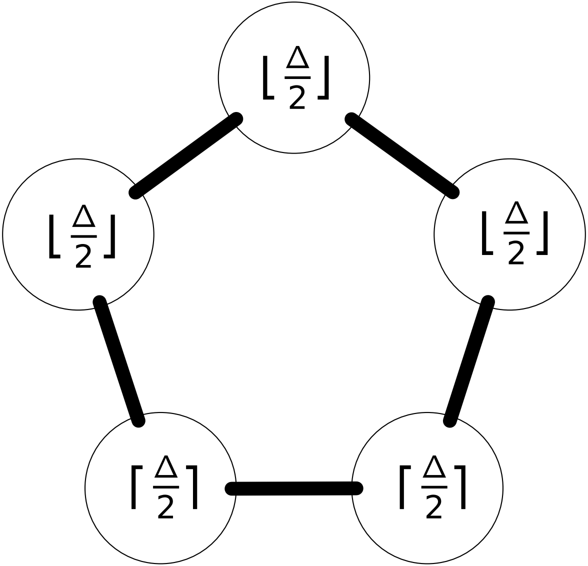
Conjecture

Conjecture (Erdős-Nešetřil '85)

For any graph G with maximum degree Δ,

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{for odd } \Delta \end{cases}$$

Blow-Up of C_5



$\lfloor \frac{\Delta}{2} \rfloor$
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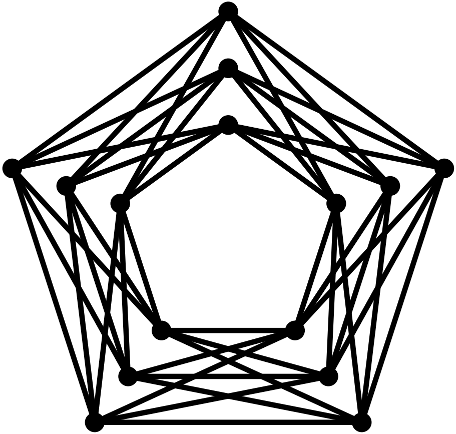

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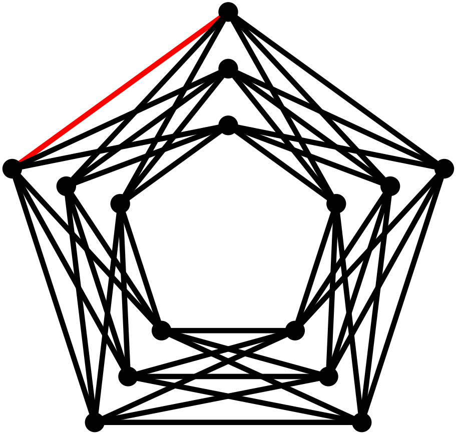

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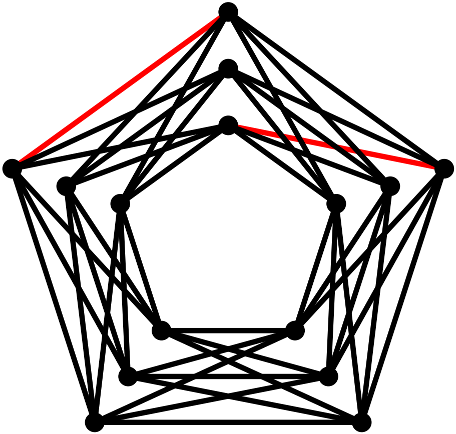

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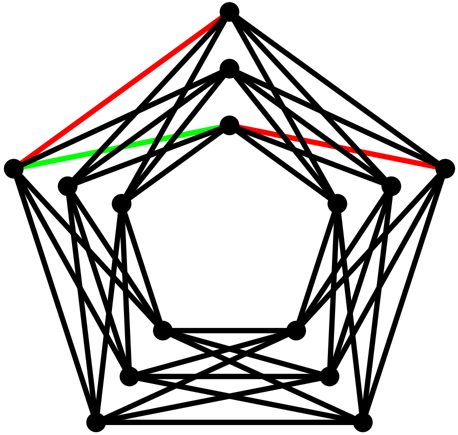

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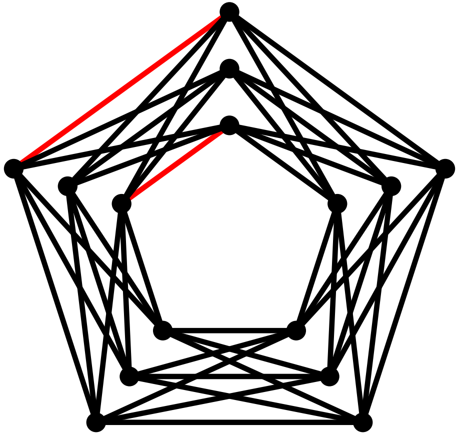

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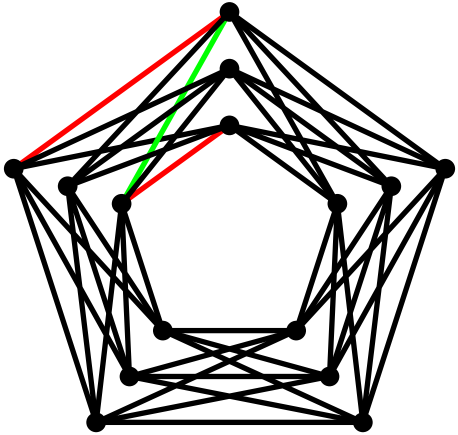

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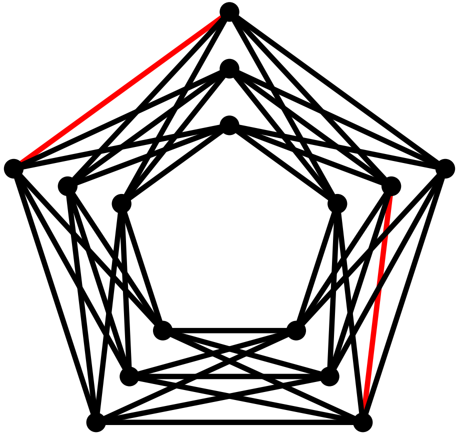

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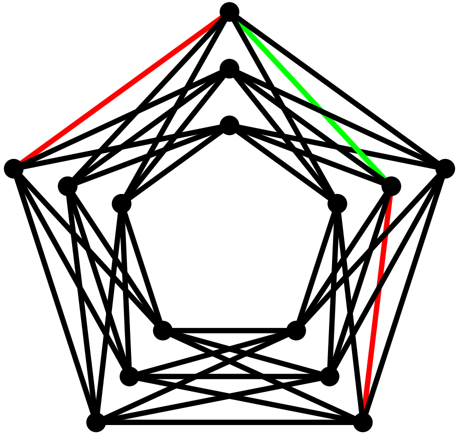

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$$\chi'_s(\text{Blow-up of } C_5) = \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{for odd } \Delta \end{cases}$$

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- If G is $(2K_2)$-free, then $\chi'_s(G) = |E(G)|$.

Theorem (Chung-Gyárfás-Trotter-Tuza '90)

The number of edges in a $(2K_2)$-free graph with max degree Δ is at most $\begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{for odd } \Delta. \end{cases}$
Additionally, the blow-up of C_5 is the unique extremal graph.

Conjecture

Conjecture (Erdős-Nešetřil '85)

For any graph G with maximum degree Δ,

$$\chi'_s(G) \le \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta \\ \frac{5}{4}\Delta^2 - \frac{1}{2}\Delta + \frac{1}{4}, & \text{for odd } \Delta \end{cases}$$

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- For $\Delta = 4$, $\chi'_s(G) \leq 21$ (Huang-S-Yu '17++)

Proof Sketch

Theorem (Huang-S-Yu '17++)

If G is a multigraph with $\Delta(G) \leq 4$, then $\chi'_s(G) \leq 21$.

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- Among all counterexamples, choose G so that $|V(G)| + |E(G)|$ is minimized.
- So $\Delta(G) \leq 4$ and $\chi'_s(G) > 21$.

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Properties of a Minimal Counterexample G

- G is 4-regular, simple, etc.

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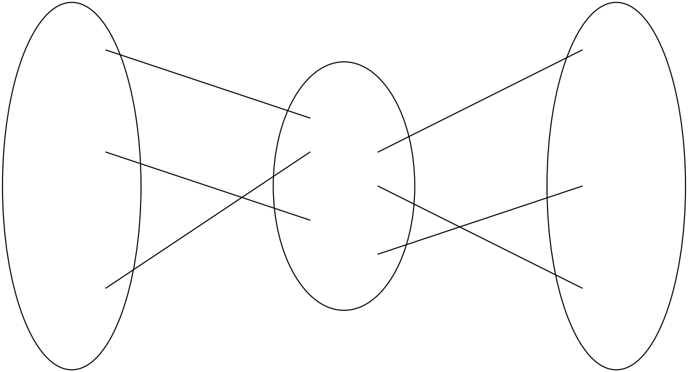
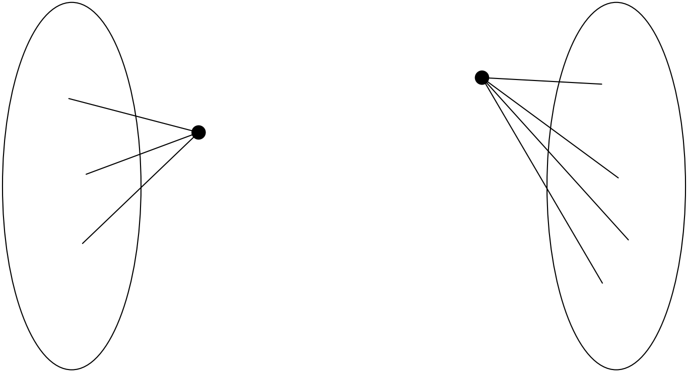
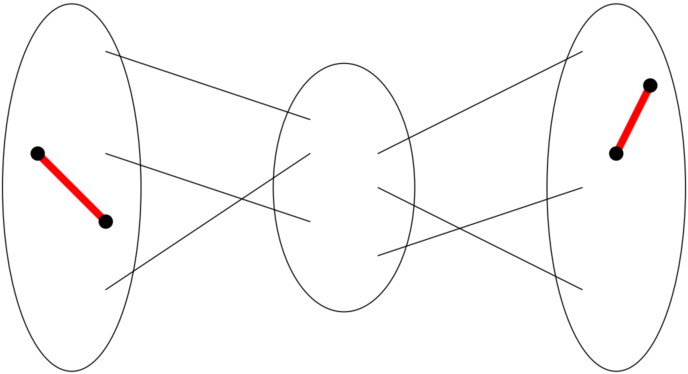
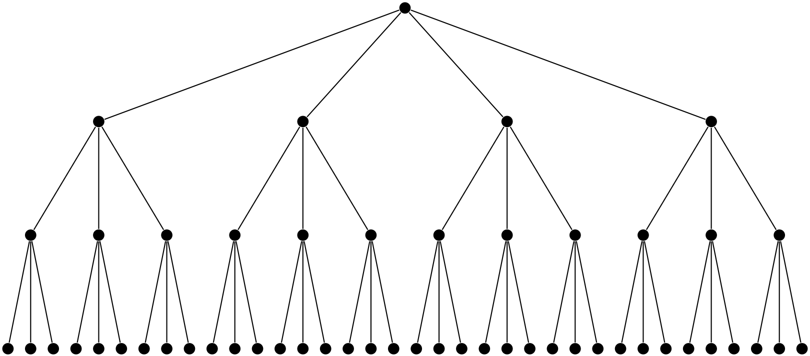
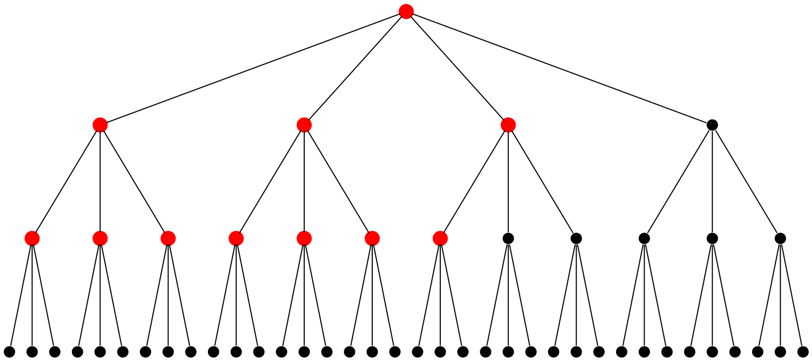
- Show that M contains some special vertices.
- Case analysis and color.

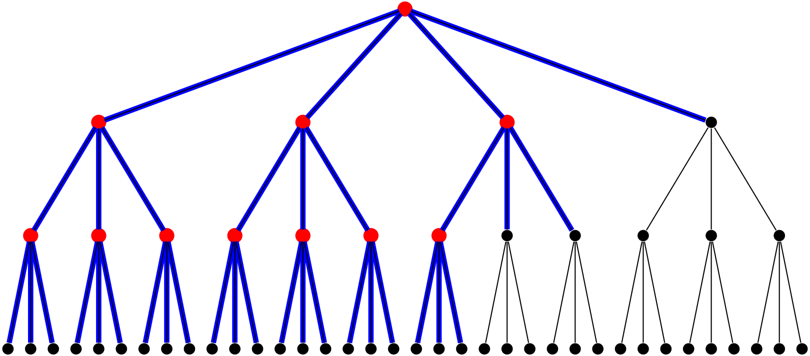












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3. If $\Delta \leq 3$ and G has 'large' girth, then $\chi'_s(G) \leq 5$

Open Problems

Conjecture (Erdős-Nešetřil '85)

If $\Delta(G) \leq 4$, then $\chi_s'(G) \leq 20$.

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3. If $\Delta \leq 3$ and G has 'large' girth, then $\chi_s'(G) \leq 5$

Theorem (Faudree et al. '90)

If G is a planar graph with maximum degree Δ, then

$$4\Delta - 4 \leq \chi_s'(G) \leq 4\Delta + 4.$$

MIGHTY LVIII

Grand Valley State University
October 6-7, 2017

Plenary Speakers:
Doug West
David Galvin

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MIGHTY_LVIII@ gvsu.edu

Thanks for your attention!

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