The Extremal Function and Colin de Verdière Parameter

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May 20, 2017

Definition

Write $V(G) = \{1, 2, \ldots, n\}$.

The Colin de Verdière parameter $\mu(G)$ of a connected graph G is the maximum corank of any symmetric real matrix M such that:

1. For all $i \neq j$, $M_{i,j} < 0$ if $ij \in E(G)$, and $M_{i,j} = 0$ if $ij \notin E(G)$.
2. M has exactly one negative eigenvalue, and it has multiplicity 1.
3. There does not exist a non-zero, symmetric, real matrix X such that $MX = 0$ and $X_{i,j} = 0$ when $ij \in E(G)$ or $i = j$.

If G has connected components $G_1, G_2, \ldots, G_m$, then define $\mu(G) := \max_{1 \leq i \leq m} \mu(G_i)$.

Some Motivation

If H is a minor of a graph G, then $\mu(H) \leq \mu(G)$.

Theorem

- $\mu(G) \leq 1 \iff$ *G subgraph of a path*
- $\mu(G) \leq 2 \iff$ *G outerplanar*
- $\mu(G) \leq 3 \iff$ *G planar*
- $\mu(G) \leq 4 \iff$ *G linklessly embeddable in $\mathbb{R}^3$*

[Colin de Verdière 90], [Robertson, Seymour, and Thomas 93,95], [Lovász and Schrijver 98]

Coloring Conjectures

Definition

The Hadwiger number $h(G)$ is the maximum integer s.t. G has $K_{h(G)}$ as a minor.

Observation

$\mu(K_t) = t - 1$, so $h(G) - 1 \leq \mu(G)$

Hadwiger's Conjecture

$\chi(G) \leq h(G)$

CDV's Coloring Conjecture

$\chi(G) \leq \mu(G) + 1$

How much weaker is CDV coloring conjecture?

Planar graphs are 4-colorable (4CC)	$\iff$	Graphs G with $\mu(G) \leq 3$ have $\chi(G) \leq \mu(G) + 1$

Theorem

4CC $\implies$ Graphs with no K_5 minor are 4-colorable. [Wagner 37]

4CC $\implies$ Graphs with no K_6 minor are 5-colorable. [Robertson, Seymour, and Thomas 93]

Extremal Function and Hadwiger's Conjecture

As a function of $h(G)$, the best known is:

Theorem

There exists an absolute constant c s.t.
$\chi(G) \leq c \cdot h(G)\sqrt{\log h(G)}$.

This is shown by average degree arguments:

Theorem

There exists an absolute constant c_1 such that
$|E(G)| \leq c_1 \cdot h(G)\sqrt{\log h(G)}|V(G)|$.

Theorem

There exists an absolute constant c_0 such that for every integer t there exists a graph G with $h(G) \geq t$ and
$|E(G)| > c_0 \cdot h(G)\sqrt{\log h(G)}|V(G)|$.

[Kostochka 82], [Thomason 84]

Small Hadwiger Number

Theorem

For $t \leq 5$, if G is a graph with $h(G) \leq t + 1$ and $|V(G)| \geq t$, then $|E(G)| \leq t|V(G)| - \binom{t+1}{2}$.

[Mader 68]

Conjecture

For all $t \in \mathbb{Z}^+$, if G is a graph with $\mu(G) \leq t$ and $|V(G)| \geq t$, then $|E(G)| \leq t|V(G)| - \binom{t+1}{2}$.

The conjecture would imply that for all graphs G, $\chi(G) \leq 2\mu(G)$!

Definition

A graph G is **chordal** if for every cycle C in G of length greater than 3, $G[V(C)]$ is not isomorphic to C.

Main Theorem

If G is a graph such that either

- $\mu(G) \le 7$, *or*
- $\mu(G) \ge |V(G)| - 6$, *or*
- G *is chordal, or*
- $\overline{G}$ *is chordal,*

then for all $t \in \mathbb{Z}^+$ with $\mu(G) \le t$ and $|V(G)| \ge t$,
$|E(G)| \le t|V(G)| - \binom{t+1}{2}$.

[RM 2017+]

Main Theorem

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$|E(G)| \leq t|V(G)| - \binom{t+1}{2}$.

[RM 2017+]

Observation

False for Hadwiger number: $K_{2,2,2,2,2}$ and $K_{r,r}$ for large enough r.

Nordhaus-Gaddum Problems

Graph complement conjecture for CDV Parameter:

Conjecture

$\mu(G) + \mu(\overline{G}) \geq |V(G)| - 2$

Theorem

True if G planar.

[Kotlov, Lovász, and Vempala 97]

Theorem

True if G chordal.

[Mitchell and Yengulalp 16]

Nordhaus-Gaddum Problems

Lemma

If G is a graph on n vertices, and $t \in \mathbb{Z}^+$ with $t \leq n$, then:
$|E(G)| \leq tn - \binom{t+1}{2} \iff |E(\overline{G})| \geq \binom{n-t}{2}$.

Theorem

For every graph G, either $|E(G)| \geq \binom{\mu(G)+1}{2}$, or $G \cong K_{3,3}$.

[Pendavingh 98]

Observation

If the GCC for CDV parameter is true, then for every graph G and every $t \in \mathbb{Z}^+$ with $t \leq |V(G)|$ and $\mu(G) \leq t$,
$|E(G)| \leq (t+1)|V(G)| - \binom{t+2}{2}$.

Proof when $\mu(G) \leq 7$

Theorem

Let G be a graph so that $\mu(G) \leq 3$. Then $\mu(G) + \mu(\overline{G}) \geq |V(G)| - 2$.

[Kotlov, Lovász, and Vempala 97]

Observation

$h(G) \leq \mu(G) + 1$

Proof.

Then $\mu(K_{2,2,2,2,2}) \geq 7$, $\mu(K_{2,2,2,3,3}) \geq 8$, and $\mu(K_{1,2,2,2,2,2}) \geq 8$.

Case: $\mu(G) \leq 5$. Then $h(G) \leq \mu(G) + 1$.

Case: $\mu(G) = 6$. Then G has no $K_{2,2,2,2,2}$ minor.

Case: $\mu(G) = 7$. Then G has no $K_{2,2,2,3,3}$ or $K_{1,2,2,2,2,2}$ minor. □

Definition

G is a **pure** k-sum of G_1 and G_2 if G can be formed by identifying a k-clique in G_1 with a k-clique in G_2.

Theorem

Let G be a graph with $h(G) \leq 7$, $|V(G)| \geq 6$, and $|E(G)| > 6|V(G)| - 21$. Then $|E(G)| = 6|V(G)| - 20$, and G can be built by pure 5-sums of $K_{2,2,2,2,2}$.

[Jørgensen 94]

Theorem

Let G be a graph with $h(G) \leq 8$, $|V(G)| \geq 7$, and $|E(G)| > 7|V(G)| - 28$. Then $|E(G)| = 7|V(G)| - 27$, and either either G is isomorphic to $K_{2,2,2,3,3}$, or G can be built by pure 6-sums of $K_{1,2,2,2,2,2}$.

[Song and Thomas 06]

Overview

Conjecture

For all $t \in \mathbb{Z}^+$*, if* G *is a graph with* $\mu(G) \leq t$ *and* $|V(G)| \geq t$*, then* $|E(G)| \leq t|V(G)| - \binom{t+1}{2}$*.*

Observation

Implies a weakening of Hadwiger's Conjecture that is as strong as the 4CC to within a factor of 2. That is, that $\chi(G) \leq 2\mu(G)$*.*

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Future Work

- If the edge upper bound conjecture is true for G_1 and G_2, is it true for their join?
- If G satisfies $\mu(G) + \mu(\overline{G}) \geq |V(G)| - 2$, then does a subdivision of G? What about a graph obtained from G by a ΔY-transform?

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