Double-critical graph conjecture for claw-free graphs

Martin Rolek
Joint work with Zi-Xia Song

University of Central Florida

Shanks Workshop: 29th Cumberland Conference on Combinatorics, Graph Theory, and Computing
Vanderbilt University
May 20, 2017

- All graphs considered have no loops and no multiple edges.

- All graphs considered have no loops and no multiple edges.
- A graph is t-**colorable** if there exists a function $c : V(G) \to \{1, \ldots, t\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E(G)$.

- All graphs considered have no loops and no multiple edges.
- A graph is t-**colorable** if there exists a function $c : V(G) \to \{1, \ldots, t\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E(G)$.
- $\chi(G) := \min\{t : G \text{ is } t\text{-colorable}\}$.

- All graphs considered have no loops and no multiple edges.
- A graph is t-**colorable** if there exists a function $c : V(G) \to \{1, \ldots, t\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E(G)$.
- $\chi(G) := \min\{t : G \text{ is } t\text{-colorable}\}$.
- A graph G is t-**chromatic** if $\chi(G) = t$.

- All graphs considered have no loops and no multiple edges.
- A graph is t-**colorable** if there exists a function $c : V(G) \to \{1, \ldots, t\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E(G)$.
- $\chi(G) := \min\{t : G \text{ is } t\text{-colorable}\}$.
- A graph G is t-**chromatic** if $\chi(G) = t$.
- $\omega(G) := \max\{t : K_t \subseteq G\}$.

- All graphs considered have no loops and no multiple edges.
- A graph is t-**colorable** if there exists a function $c : V(G) \to \{1, \dots, t\}$ such that $c(u) \neq c(v)$ for all edges $uv \in E(G)$.
- $\chi(G) := \min\{t : G \text{ is } t\text{-colorable}\}$.
- A graph G is t-**chromatic** if $\chi(G) = t$.
- $\omega(G) := \max\{t : K_t \subseteq G\}$.
- $\chi(G) \geq \omega(G)$, and equality holds for many graphs, i.e. perfect graphs.

Question - Erdős (1968)

If $\chi(G) = 3t$, then it is trivial to show G contains t vertex-disjoint odd cycles.
Is it true for $t \geq 2$ that every $(3t-1)$-chromatic critical graph with sufficiently many vertices contains t vertex-disjoint odd cycles?

Question - Erdős (1968)

If $\chi(G) = 3t$, then it is trivial to show G contains t vertex-disjoint odd cycles.
Is it true for $t \geq 2$ that every $(3t-1)$-chromatic critical graph with sufficiently many vertices contains t vertex-disjoint odd cycles?

- $(3t-2)$-chromatic critical graphs may not have t vertex-disjoint odd cycles (Gallai 1968, $t = 2$)

Question - Erdős (1968)

If $\chi(G) = 3t$, then it is trivial to show G contains t vertex-disjoint odd cycles.
Is it true for $t \geq 2$ that every $(3t-1)$-chromatic critical graph with sufficiently many vertices contains t vertex-disjoint odd cycles?

- $(3t-2)$-chromatic critical graphs may not have t vertex-disjoint odd cycles (Gallai 1968, $t = 2$)
- Is it true that every 5-chromatic critical graph with sufficiently many vertices contains two vertex-disjoint odd cycles?

Question - Erdős (1968)

If $\chi(G) = 3t$, then it is trivial to show G contains t vertex-disjoint odd cycles.
Is it true for $t \geq 2$ that every $(3t-1)$-chromatic critical graph with sufficiently many vertices contains t vertex-disjoint odd cycles?

- $(3t-2)$-chromatic critical graphs may not have t vertex-disjoint odd cycles (Gallai 1968, $t = 2$)
- Is it true that every 5-chromatic critical graph with sufficiently many vertices contains two vertex-disjoint odd cycles?
- If a 5-chromatic graph contains two vertex disjoint odd cycles, then it has two disjoint 3-chromatic subgraphs.

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)
- True for $(s, t) = (2, 4)$ (Mozhan 1987; Stiebitz 1987)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)
- True for $(s, t) = (2, 4)$ (Mozhan 1987; Stiebitz 1987)
- True for $(s, t) = (3, 3), (3, 4), (3, 5)$ (Stiebitz 1987)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)
- True for $(s, t) = (2, 4)$ (Mozhan 1987; Stiebitz 1987)
- True for $(s, t) = (3, 3), (3, 4), (3, 5)$ (Stiebitz 1987)
- True for line graphs (Kostochka and Stiebitz 2008)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)
- True for $(s, t) = (2, 4)$ (Mozhan 1987; Stiebitz 1987)
- True for $(s, t) = (3, 3), (3, 4), (3, 5)$ (Stiebitz 1987)
- True for line graphs (Kostochka and Stiebitz 2008)
- True for quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Balogh, Kostochka, Prince, and Stiebitz 2009)

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- True for $(s, t) = (2, 2)$ (easy)
- True for $(s, t) = (2, 3)$ (Brown and Jung 1969)
- True for $(s, t) = (2, 4)$ (Mozhan 1987; Stiebitz 1987)
- True for $(s, t) = (3, 3), (3, 4), (3, 5)$ (Stiebitz 1987)
- True for line graphs (Kostochka and Stiebitz 2008)
- True for quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Balogh, Kostochka, Prince, and Stiebitz 2009)
- Otherwise fairly wide open

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- If we fix $s = 2$, the conjecture claims there exists an edge uv such that $\chi(G - u - v) \geq \chi(G) - 1$.

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- If we fix $s = 2$, the conjecture claims there exists an edge uv such that $\chi(G - u - v) \geq \chi(G) - 1$.
- A connected graph G is **double-critical** if for every edge $uv \in E(G)$, $\chi(G - u - v) = \chi(G) - 2$.

Erdős-Lovász Tihany Conjecture (1968)

Let G be a graph with $\chi(G) > \omega(G)$, and let $s, t \geq 2$ be integers such that $\chi(G) = s + t - 1$. Then G contains two disjoint subgraphs H_1 and H_2 such that $\chi(H_1) \geq s$ and $\chi(H_2) \geq t$.

- If we fix $s = 2$, the conjecture claims there exists an edge uv such that $\chi(G - u - v) \geq \chi(G) - 1$.
- A connected graph G is **double-critical** if for every edge $uv \in E(G)$, $\chi(G - u - v) = \chi(G) - 2$.

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

- Special case $s = 2$ of Erdős-Lovász Tihany Conjecture.

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

- Special case $s = 2$ of Erdős-Lovász Tihany Conjecture.
- True for line graphs, quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Kostochka and Stiebitz 2008; Balogh, Kostochka, Prince, and Stiebitz 2009)

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

- Special case $s = 2$ of Erdős-Lovász Tihany Conjecture.
- True for line graphs, quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Kostochka and Stiebitz 2008; Balogh, Kostochka, Prince, and Stiebitz 2009)
- True for $t \leq 5$ (Brown and Jung 1969; Mozhan 1987; Stiebitz 1987)

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

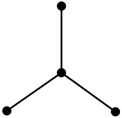
- Special case $s = 2$ of Erdős-Lovász Tihany Conjecture.
- True for line graphs, quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Kostochka and Stiebitz 2008; Balogh, Kostochka, Prince, and Stiebitz 2009)
- True for $t \leq 5$ (Brown and Jung 1969; Mozhan 1987; Stiebitz 1987)
- Open for $t \geq 6$.

Double-Critical Graph Conjecture (Erdős and Lovász 1968)

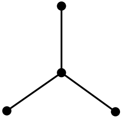
For every $t \geq 1$, the only double-critical, t-chromatic graph is K_t.

- Special case $s = 2$ of Erdős-Lovász Tihany Conjecture.
- True for line graphs, quasi-line graphs, and true for graphs with $\alpha(G) = 2$ (Kostochka and Stiebitz 2008; Balogh, Kostochka, Prince, and Stiebitz 2009)
- True for $t \leq 5$ (Brown and Jung 1969; Mozhan 1987; Stiebitz 1987)
- Open for $t \geq 6$.
- Not yet known if a double critical, t-chromatic graph $G \neq K_t$ contains a K_4 subgraph for $t \geq 6$.

- A graph is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph.



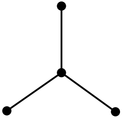
- A graph is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph.



Theorem - Huang and Yu (2016+)

If G is a claw-free, double-critical, 6-chromatic graph, then $G = K_6$.

- A graph is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph.



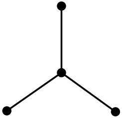
Theorem - Huang and Yu (2016+)

If G is a claw-free, double-critical, 6-chromatic graph, then $G = K_6$.

Theorem - R. and Song (2017)

For $t \in \{6, 7, 8\}$, if G is a claw-free, double-critical, t-chromatic graph, then $G = K_t$.

- A graph is **claw-free** if it does not contain $K_{1,3}$ as an induced subgraph.



Theorem - Huang and Yu (2016+)

If G is a claw-free, double-critical, 6-chromatic graph, then $G = K_6$.

Theorem - R. and Song (2017)

For $t \in \{6, 7, 8\}$, if G is a claw-free, double-critical, t-chromatic graph, then $G = K_t$.

- For $t = 6$, our proof is different and shorter.

Proposition (Kawarabayashi, Pedersen, and Toft 2011)

Suppose G is a non-complete, double-critical, t-chromatic graph. Then the following are true:

- $\delta(G) \geq t + 1$.
- Every edge belongs to at least $t - 2$ triangles.
- If $x \in V(G)$ such that $d(x) < |V(G)| - 1$, then $\chi(G[N(x)]) \leq t - 3$.
- If $d(x) = t + 1$, then $\overline{G[N(x)]}$ consists only of isolated vertices and/or disjoint cycles of length at least 5.

Theorem - Kawarabayashi, Pedersen, and Toft (2010)

If G is a non-complete, double-critical, t-chromatic graph, then no two vertices of degree $t+1$ are adjacent.

Theorem - Kawarabayashi, Pedersen, and Toft (2010)

If G is a non-complete, double-critical, t-chromatic graph, then no two vertices of degree $t+1$ are adjacent.

Theorem - R. and Song (2017)

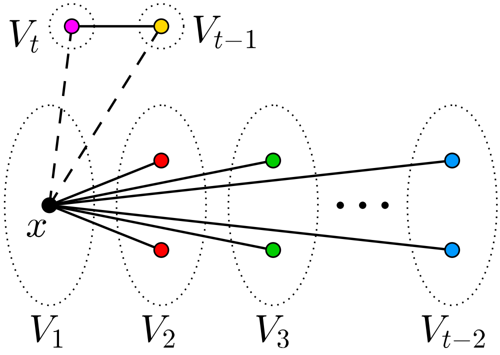
If G is a non-complete, double-critical, t-chromatic graph, then no vertex of degree $t+1$ is adjacent to any vertex of degree $\leq t+3$.

Proposition - R. and Song (2017)

If G is a non-complete, double-critical, t-chromatic claw-free graph, then for any $x \in V(G)$, $d(x) \leq 2t - 4$. Furthermore, if $d(x) < |V(G)| - 1$, then $d(x) \leq 2t - 6$.

Proposition - R. and Song (2017)

If G is a non-complete, double-critical, t-chromatic claw-free graph, then for any $x \in V(G)$, $d(x) \leq 2t - 4$. Furthermore, if $d(x) < |V(G)| - 1$, then $d(x) \leq 2t - 6$.



Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

- Suppose G is claw-free, double-critical, t-chromatic and $G \neq K_t$, and let $x \in V(G)$ such that $d(x) = \delta(x)$.

Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

- Suppose G is claw-free, double-critical, t-chromatic and $G \neq K_t$, and let $x \in V(G)$ such that $d(x) = \delta(x)$.
- $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.

Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

- Suppose G is claw-free, double-critical, t-chromatic and $G \neq K_t$, and let $x \in V(G)$ such that $d(x) = \delta(x)$.
- $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.
- If $t = 7$, then G is 8-regular, contradicting that vertices of degree $t + 1$ are not adjacent.

Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

- Suppose G is claw-free, double-critical, t-chromatic and $G \neq K_t$, and let $x \in V(G)$ such that $d(x) = \delta(x)$.
- $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.
- If $t = 7$, then G is 8-regular, contradicting that vertices of degree $t + 1$ are not adjacent.
- If $t = 8$, then no vertex of degree 9 is adjacent to a vertex of degree 9 or 10. Hence, G is 10-regular.

Theorem (R. and Song 2017)

If G is a claw-free, double-critical, t-chromatic graph for $t \in \{6, 7, 8\}$, then $G = K_t$.

Proof sketch.

- Suppose G is claw-free, double-critical, t-chromatic and $G \neq K_t$, and let $x \in V(G)$ such that $d(x) = \delta(x)$.
- $t + 1 \leq d(x) \leq 2t - 6$, and so $t \geq 7$.
- If $t = 7$, then G is 8-regular, contradicting that vertices of degree $t + 1$ are not adjacent.
- If $t = 8$, then no vertex of degree 9 is adjacent to a vertex of degree 9 or 10. Hence, G is 10-regular.
- By examining the structure of $N(x)$, we find a claw.

Question

Is it true that the only claw-free, double-critical, 9-chromatic graph is K_9? Is it true that any claw-free, double-critical, t-chromatic graph is K_t for all t?

Question

Is it true that the only claw-free, double-critical, 9-chromatic graph is K_9? Is it true that any claw-free, double-critical, t-chromatic graph is K_t for all t?

- For the case $t = 9$, our results show that any such graph G must contain only of vertices of degrees 11 and 12.

Question

Is it true that the only claw-free, double-critical, 9-chromatic graph is K_9? Is it true that any claw-free, double-critical, t-chromatic graph is K_t for all t?

- For the case $t = 9$, our results show that any such graph G must contain only of vertices of degrees 11 and 12.

Question

In a double-critical, t chromatic graph, no vertex of degree $t + i$ is adjacent to a vertex of $t + j$ for $(i, j) \in \{(1, 1), (1, 2), (1, 3)\}$. Can any other pairs be found? $(2, 2)$? $(2, 3)$?

Question

Is it true that the only claw-free, double-critical, 9-chromatic graph is K_9? Is it true that any claw-free, double-critical, t-chromatic graph is K_t for all t?

- For the case $t = 9$, our results show that any such graph G must contain only of vertices of degrees 11 and 12.

Question

In a double-critical, t chromatic graph, no vertex of degree $t + i$ is adjacent to a vertex of $t + j$ for $(i, j) \in \{(1, 1), (1, 2), (1, 3)\}$. Can any other pairs be found? $(2, 2)$? $(2, 3)$?

Question

Is it true that the only double-critical, 6-chromatic graph is K_6?

Thank you!