Conditional Connectivity in Networks

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Outline

Networks

Interconnection networks play an important role in parallel and distributed computing/communication systems and data centers. An interconnection network can be modeled by a graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network.

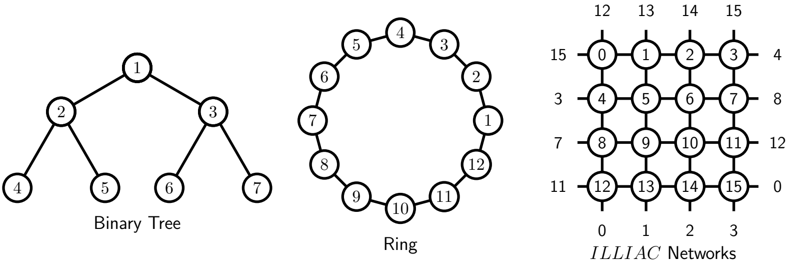
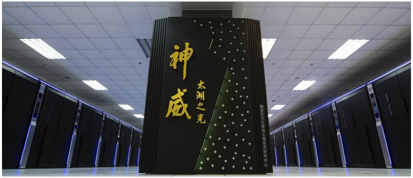


Figure: Topological structure of some simple networks

Sunway TaihuLight Supercomputer (Top 1)



Architecture and Performance

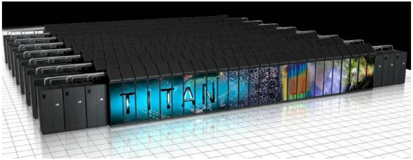
- Computer nodes 40,960
- Number of core 10,649,600
- Total CPU plus coprocessor memory 1.31 PB
- Total peak performance 93 petaflops.

Tianhe-2 Supercomputer (Top 2)

Architecture and Performance

- Computer nodes 16,000
- Number of core 3,120,000
- Total CPU plus coprocessor memory 1,375 TB
- Total peak performance 33.9 petaflops.

Titan supercomputer (Top 3)



Architecture and Performance

- Computer nodes 18,688
- Number of core 299,008
- Total CPU plus coprocessor memory 710 TB
- Total peak performance 20 petaflops.

Characteristics of interconnection networks

Extendability

It should be possible to build a network of any given size, or at least to build arbitrarily large versions of the network. Furthermore, it would be easy to construct large networks from small ones.

Symmetry

Regularity and some symmetric properties on the graph.

Classical connectivity

Connectivity, Edge connectivity

A subset $S \subset V(G)(S \subset E(G))$ of a connected graph G is called a *cut(edge-cut)* if $G - S$ is disconnected. The *connectivity(edge-connectivity)* $\kappa(G)(\lambda(G))$ of G is defined as the minimum cardinality over all cuts(edge-cuts) of G, that is

$$\kappa(G) = \min\{|S| : \ S \textit{ is a cut of } G\ \},$$

$$\lambda(G) = \min\{|S| : \ S \textit{ is an edge-cut of } G\ \}.$$

Classical connectivity

Flaws

When computing these parameters, one implicitly assumes that all links incident with the same processor may fail simultaneously. Consequently, this measurement is inaccurate for large-scale processing systems in which some subsets of system components can not fail at the same time in real applications.

Conditional connectivity

Definition(Harary, 1983)

The conditional connectivity of G with respect to some property P is the smallest cardinality of a set S of vertices, if any, such that every component of the disconnected graph $G - S$ has property P.

- In 1989, Esfahanian proposed restricted connectivity
- In 1994, Latifi generalized it to the restricted h-connectivity

Restricted h-connectivity(Latifi *et al.* 1994)

$\kappa^{(h)}(G), \lambda^{(h)}(G)$

For a given integer $h\,(\geq 0)$, a vertex(edge) subset S of a connected graph G is called an h-*cut*(h-*edge-cut*), if $G-S$ is disconnected and has the minimum degree $\delta(G-S) \geq h$. The h-*super connectivity*(*edge connectivity*) of G, denoted by $\kappa^{(h)}(G)(\lambda^{(h)}(G))$, is defined as the minimum cardinality over all . That is

$$\kappa^{(h)}(G) = \min\{|S| : \ S \textit{ is an } h\textit{-cut of } G \ \}.$$

$$\lambda^{(h)}(G) = \min\{|S| : \ S \textit{ is an } h\textit{-edge-cut of } G \ \}.$$

Complexity

Complexity (Oh, Choi, and Esfahanian, 1991)

The problem of finding the least cardinality S such that S is a conditional cut of G is NP-complete.

Cayley graphs

$Cay(\Gamma, S)$

$\Gamma = (Z, \circ)$ is a finite group, S is a nonempty subset of Z without identity. Cayley digraph $Cay(\Gamma, S)$ is a digraph with vertices Γ and edges $E(Cay(\Gamma, S)) = \{uv : v = u \circ s, u \in \Gamma, s \in S\}$. $S^{-1} = \{s^{-1} : \ s \in S\} = S$, $Cay(\Gamma, S)$ is an undirected graph.

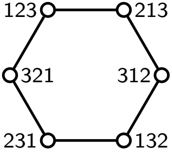
$Cay(Sym(n), T)$

$Sym(n)$ is the symmetric group on $\{1, 2, \ldots, n\}$ and T is a set of transposition of $Sym(n)$.
$G(T)$ be the graph on n vertices $\{1, 2, \ldots, n\}$ such that there is an edge ij in $G(T)$ if and only if transposition $(ij) \in T$.

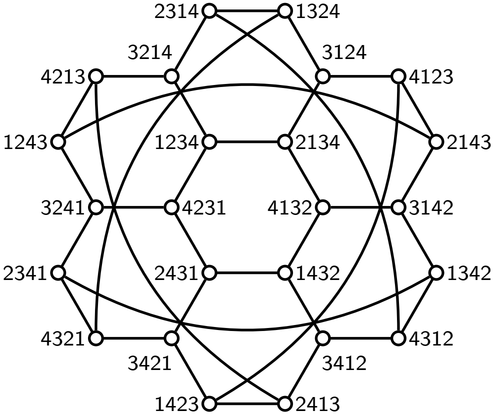
If $G(T)$ is a star, $Cay(Sym(n), T)$ be star graph; if $G(T)$ is a path, $Cay(Sym(n), T)$ be bubble-sort graph.

Star graphs S_2, S_3, S_4

S_2



S_3



S_4

Hierarchical Structure

Use $S_n^{j:i}$ to denote the subgraph of S_n induced by all vertices with symbol i in the j-th position, I'_n be a set $\{2, \ldots, n\}$.

The first structural property (Akers and Krishnamurthy, 1989)

For a fixed dimension $j \in I'_n$, S_n can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $i \in I_n$. Moreover, there are $(n-2)!$ independent edges between $S_n^{j:i_1}$ and $S_n^{j:i_2}$ for any $i_1, i_2 \in I_n$ with $i_1 \neq i_2$.

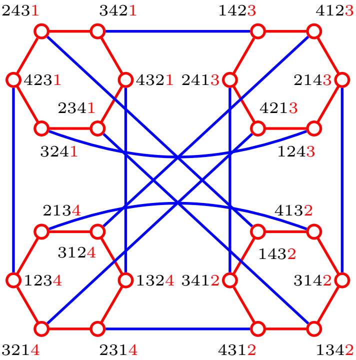
Hierarchical Structure

Use $S_n^{j:i}$ to denote the subgraph of S_n induced by all vertices with symbol i in the j-th position, I'_n be a set $\{2, \ldots, n\}$.

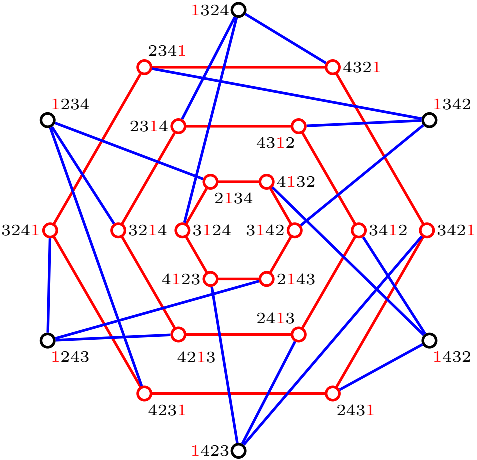
The second structural property (Shi *et al.*, 2012)

For a fixed symbol $i \in I_n$, S_n can be partitioned into n subgraphs $S_n^{j:i}$, which is isomorphic to S_{n-1} for each $j \in I'_n$ and $S_n^{1:i}$ is an independent vertex set of size $(n-1)!$. Moreover, there are a perfect matching between $S_n^{1:i}$ and $S_n^{j:i}$ for any $j \in I'_n$, and there are no edges between $S_n^{j_1:i}$ and $S_n^{j_2:i}$ for any $j_1, j_2 \in I'_n$ with $j_1 \neq j_2$.

Two structures of S_4



Partition along dimension 4



Partition along symbol 1

Some results

Theorem (Rouskov et al., 1996)

If $n \geq 3$, then $\kappa^{(1)}(S_n) = 2n - 4$.

Theorem (Wan and Zhang, 2009)

If $n \geq 4$, then $\kappa^{(2)}(S_n) = 6(n-3)$.

Conjecture (Wan and Zhang, 2009)

If $h \leq n-2$, Then $\kappa^{(h)}(S_n) = (h+1)!(n-h-1)$.

Theorem (Yang et al., 2010)

If $n \geq 4$, then $\lambda^{(2)}(S_n) = 6(n-3)$.

Strategy

For a subset $X \subseteq V(S_n)$ and $j \in I_n$, we use U_j^X to denote the set of symbols in the j-th position of vertices in X, formally, $U_j^X = \{p_j :\ p_1 \dots p_j \dots p_n \in X\}$.

Lemma (Li and Xu, 2014)

Let H be a subgraph of S_n with vertex-set X. For a fixed $h \in I_{n-1}$, if $\delta(H) \geq h$, then there exists some $j \in I'_n$ such that $|U_j^X| \geq h+1$.

Theorem (Li and Xu, 2014)

If $0 \leq h \leq n-2$, then $\kappa^{(h)}(S_n) \geq (h+1)!(n-h-1)$, $\lambda^{(h)}(S_n) \geq (h+1)!(n-h-1)$.

Some results

Theorem (Li and Xu, 2014)

If $0 \le h \le n-2$, then $\kappa^{(h)}(S_n) = \lambda^{(h)}(S_n) = (h+1)!(n-h-1)$.

Conjecture (Wan and Zhang, 2009)

If $h \le n-2$, Then $\kappa^{(h)}(S_n) = (h+1)!(n-h-1)$.

The Conjecture is proved to be correct, can we say anything more?

(n,k)-Star, A generalization of S_n

Definition (Akers and Krishnamurthy, 1989)

The n-dimensional star graph S_n has vertex-set $P(n)$ and has an edge between any two vertices if and only if one can be obtained from the other by swapping the 1-th digit and the i-th digit for $i \in I'_n$, that is, two vertices $x = p_1p_2\ldots p_i\ldots p_n$ and y are adjacent if and only if $y = p_ip_2\ldots p_{i-1}p_1p_{i+1}\ldots p_n$ for some $i \in I'_n$.

Definition (Chiang *et al.*, 1995)

An (n,k)-star graph $S_{n,k}$ is a graph with vertex-set $P(n,k)$, a vertex $p = p_1p_2\ldots p_i\ldots p_k$ being linked a vertex q if and only if q is
$\text{(a)}\ p_ip_2\cdots p_{i-1}p_1p_{i+1}\cdots p_k$, where $i \in I'_k$ (swap p_1 with p_i), or
$\text{(b)}\ p'_1p_2p_3\cdots p_k$, where $p'_1 \in I_n \setminus \{p_i :\ i \in I_k\}$ (replace p_1 by p'_1).

An Useful Tool

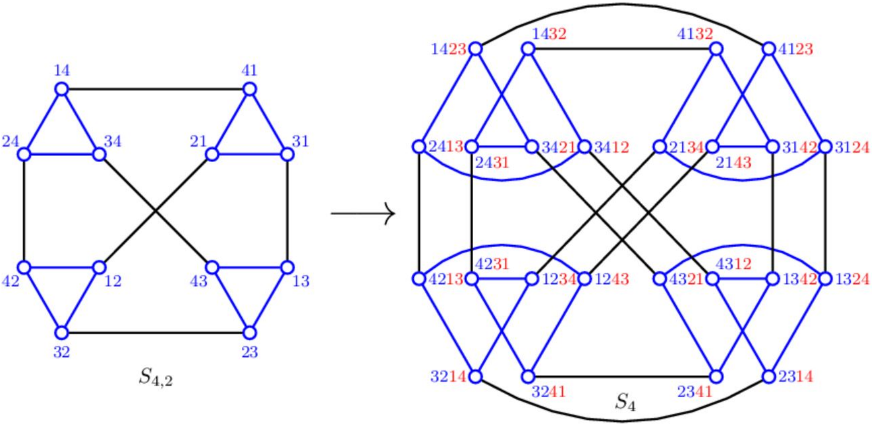
Definition, t-Split

A *t-split* graph G^t of G is a graph obtained from G by replacing each vertex x by a set V_x of t independent vertices, and replacing each edge $e = xy$ by a perfect matching E_e between V_x and V_y.

Lemma

Let G be a connected graph and G^t be a t-split graph of G. Then $\kappa^{(h)}(G^t) \leq t\,\kappa^{(h)}(G)$ and $\lambda^{(h)}(G^t) \leq t\,\lambda^{(h)}(G)$.

$(4,2)$-star graph $S_{4,2}$ and its 2-split graph



Relationship between S_n and $S_{n,k}$

Theorem (Li and Xu, 2017+)

For any k with $2 \leq k \leq n-1$, there is an $(n-k)!$-split graph of $S_{n,k}$ that is isomorphic to a star graph S_n.

Theorem (Li and Xu, 2017+)

For $2 \leq k \leq n-1$ and $n-k \leq h \leq n-2$,

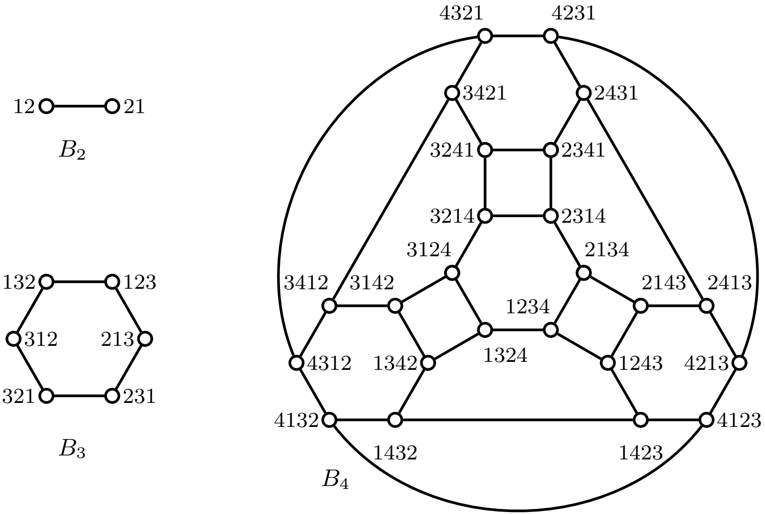
$$\kappa^{(h)}(S_{n,k}) = \lambda^{(h)}(S_{n,k}) = \frac{(h+1)!(n-h-1)}{(n-k)!}.$$

Bubble-sort graphs

Definition (Akers and Krishnamurthy, 1989)

The n-dimensional bubble-sort graph B_n has $n!$ vertices labeled by distinct permutations on $\{1, 2, ..., n\}$, and has an edge between any two vertices if and only if one can be obtained from the other by swapping the i-th digit and the $(i+1)$-th digit where $1 \leq i \leq n-1$.

Bubble graphs B_2, B_3, B_4



Some Results

Theorem (Akers and Krishnamurthy, 1989)

(1) B_n has regular degree $n-1$;
(2) $\kappa(B_n)=\lambda(B_n)=n-1$;

Structure (Akers and Krishnamurthy, 1989)

For a fixed $t\in\{1,n\}$, B_n can be partitioned into n subgraphs $B_n^{t:j}$ isomorphic to B_{n-1} for each $j\in I_n$, moreover, there are $(n-2)!$ independent edges between $B_n^{t:j_1}$ and $B_n^{t:j_2}$ for any $j_1,j_2\in I_n$ with $j_1\neq j_2$.

Theorem (Yang *et al.*, 2010)

If $n\geq 3$, then $\kappa^1(B_n)=2n-4$; if $n\geq 4$, then $\kappa^2(B_n)=4n-12$.

How about $\lambda^h(B_n),\kappa^h(B_n)$ for more bigger h?

Motivation and Methods

- Construction of the upper bound;
- The structure of B_n;
- How an h-cut(edge cut) of large networks distributed in the small B_{n-1};
- Prove the lower bound using the $(h-1)$-supper connectivity of B_{n-1}.

Fault tolerance in B_n

Theorem (Li and Xu, 2017+)

$\kappa^h(B_n) = \lambda^h(B_n) = 2^h(n-1-h)$ for any h with $2h \leq n$.

Corollary

If $n \geq 3$, then $\kappa^1(B_n) = 2n-4$; if $n \geq 4$, then $\kappa^2(B_n) = 4n-12$.

Problems

Problem

How about the h-super connectivity (edge connectivity) of B_n for $h \geq 2k+1$?

h-Atom

For a given integer $h\,(\geq 0)$, a vertex(edge) subset S of a connected graph G is called an S *be an* h*-cut(*h*-edge-cut)*, of G, the minimum connected component of $G - S$ is an h-atom(edge atom) of G.

Observation and Problems

The h-atom of S_n,Q_n is regular, How about h-atom for general Cayley graphs or regular graphs.

Thanks For Your Attention !