# Matchings, Covers, and Network Games

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29th Cumberland Conference on
Combinatorics, Graph Theory and Computing

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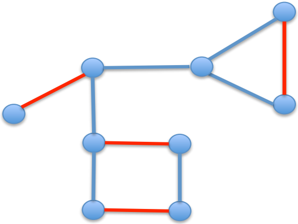

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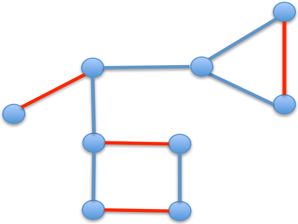

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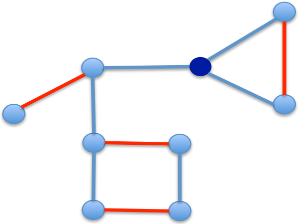

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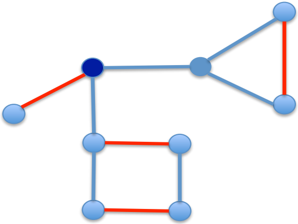

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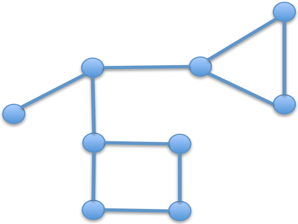

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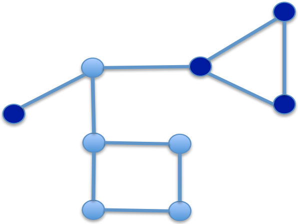

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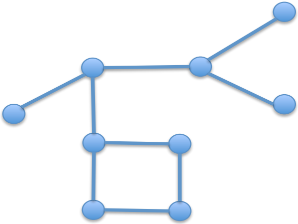

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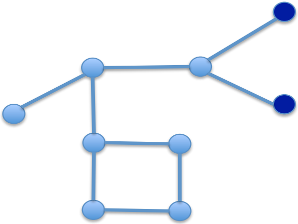

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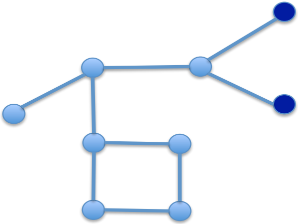

- Stable graph → Why are these graphs interesting?

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- An outcome for the game is a pair $(M, y)$

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$\rightarrow$ Let's look at this question from a graph theory perspective

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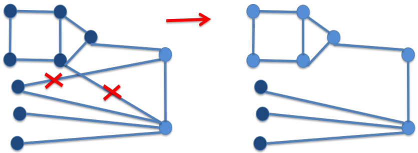

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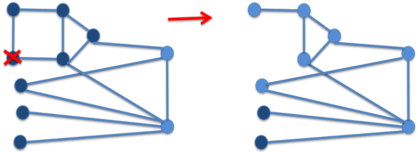

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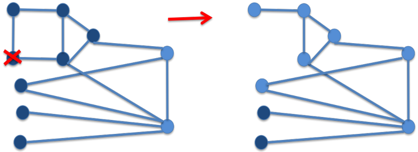

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**Combinatorial question:** *Can we efficiently find (edge-/vertex-) stabilizers of minimum cardinality?*

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$\rightarrow$ How are these results proved?

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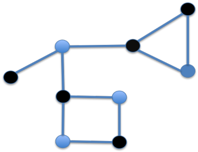

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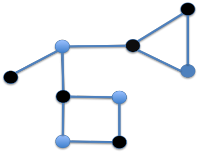

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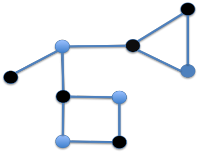

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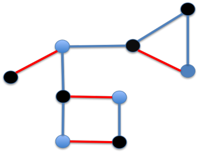

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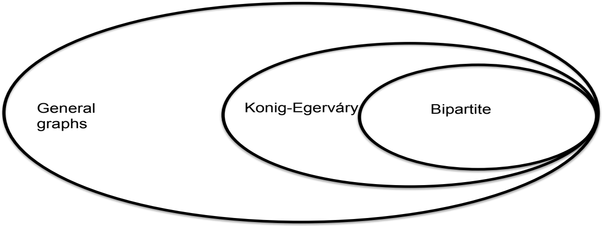

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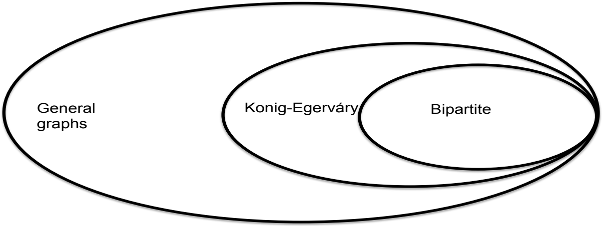


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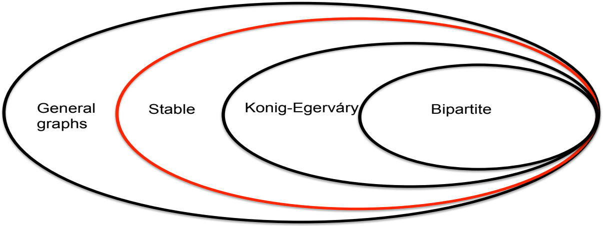


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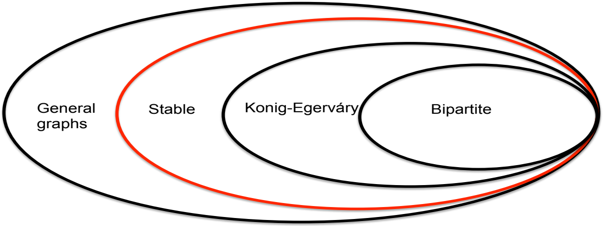


• Stable graphs are a superclass of König-Egerváry graphs, and can be characterized in terms of fractional matchings and covers.

# Fractional matchings and covers

## Fractional matchings and covers

• Finding a maximum matching of a graph $G = (V, E)$ can be formulated as the following Integer Program (IP):

$$\nu(G) := \max\{\mathbf{1}^T x : x(\delta(v)) \leq 1 \ \forall v \in V, \ x \in \{0,1\}^E\}$$

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- Feasible solutions to these LPs yield *fractional* matchings and covers!

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- **By duality:** we know that the following chain of inequalities holds for all $G$:

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G)$$

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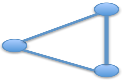
- **Example:**

$$\nu(G) \leq \nu_f(G) = \tau_f(G) \leq \tau(G)$$



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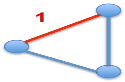


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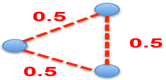


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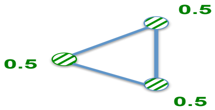


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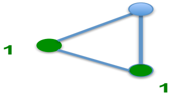


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Key ingredient: Edmonds-Gallai Decomposition of a graph.

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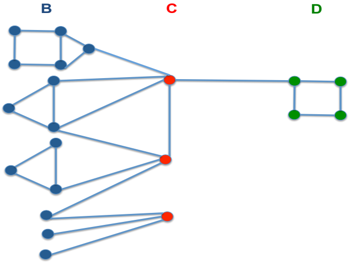

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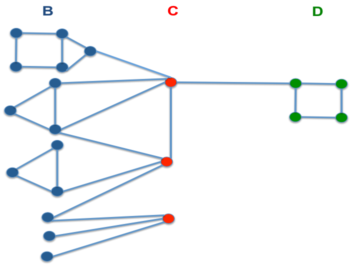


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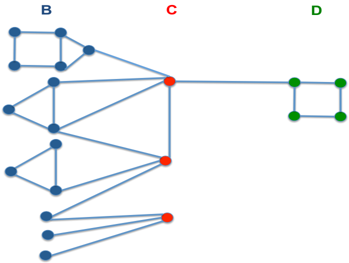


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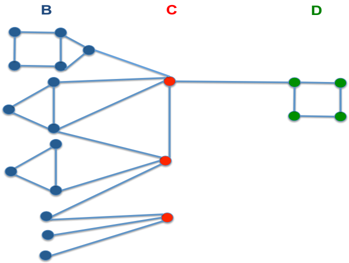


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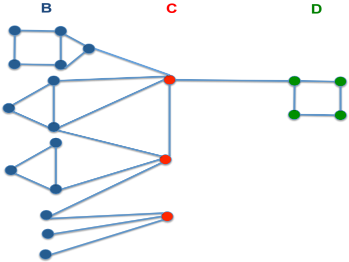


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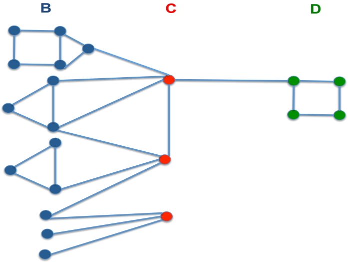


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• What is the relation between this decomposition and max matchings?

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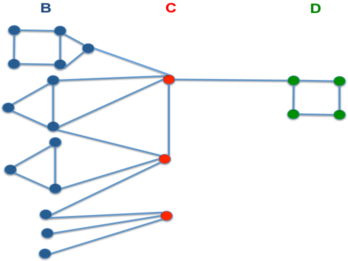

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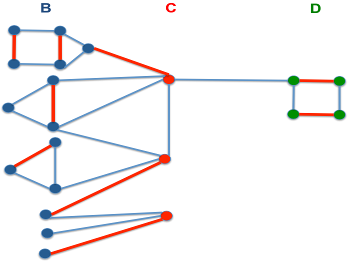

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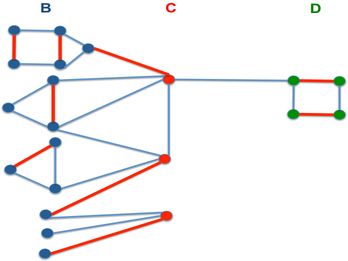

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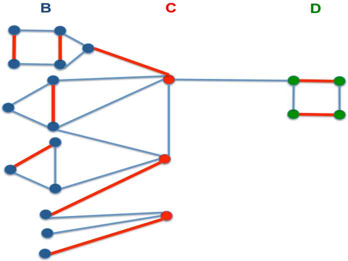


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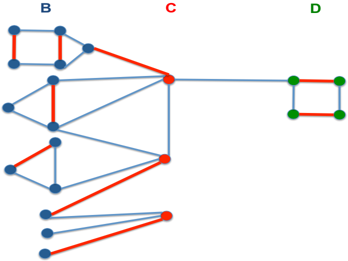


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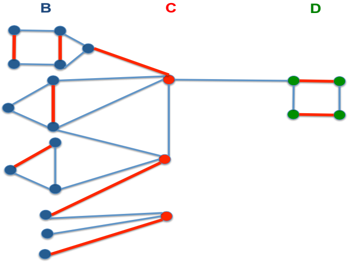


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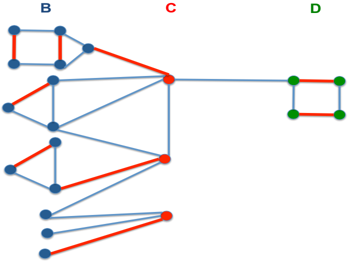


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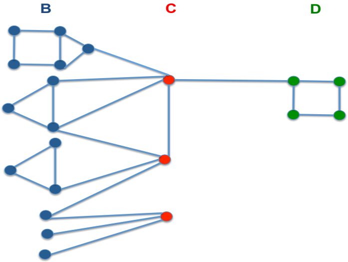

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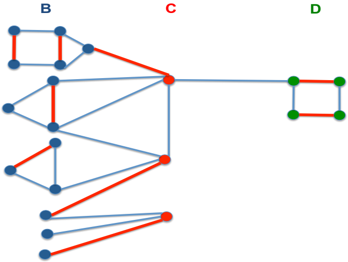

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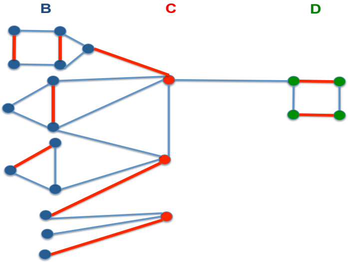

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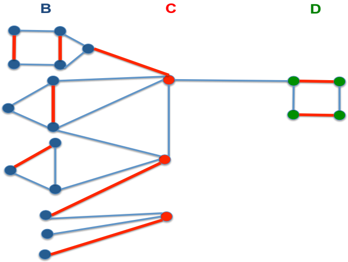


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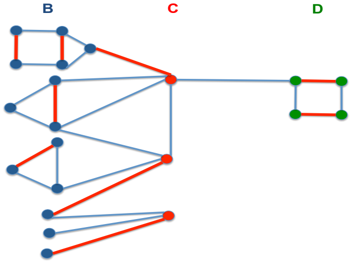


Construct a *fractional* matching $x \in \mathbb{R}^E$ as follows:

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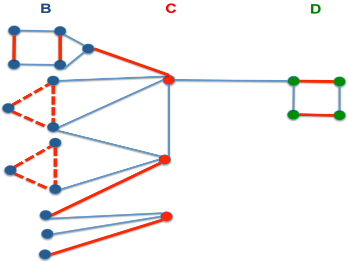


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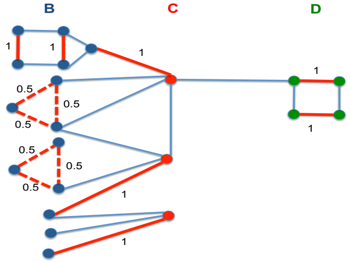


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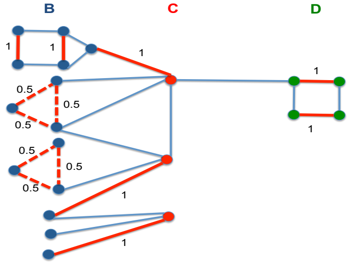


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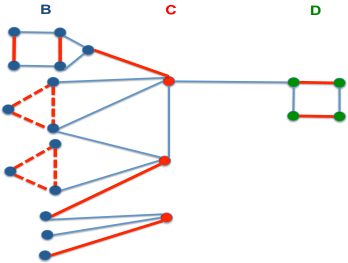

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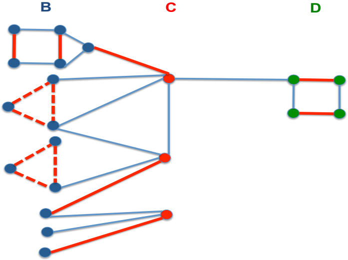

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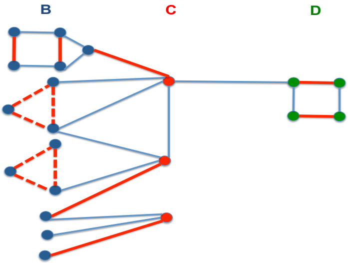


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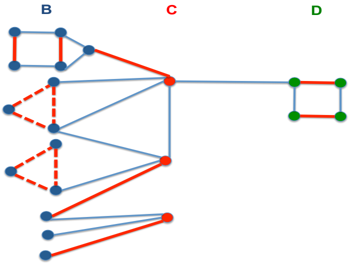


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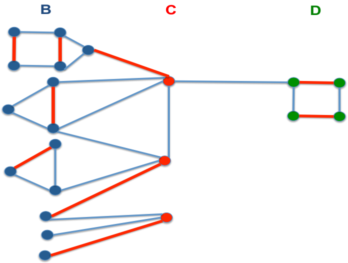


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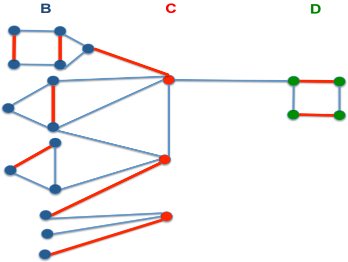

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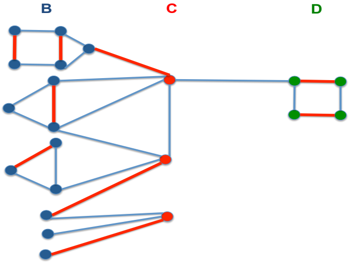

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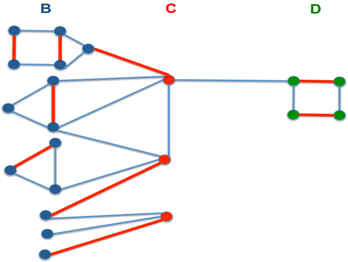


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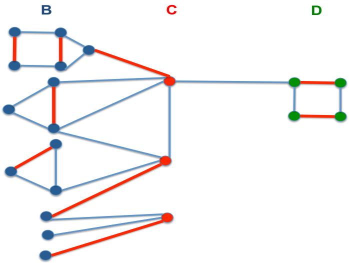


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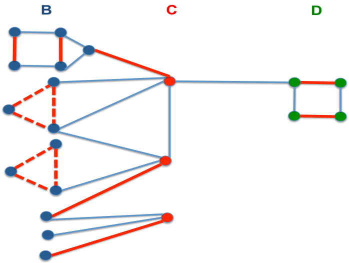


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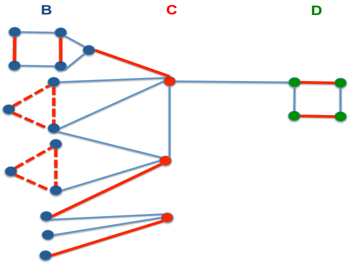


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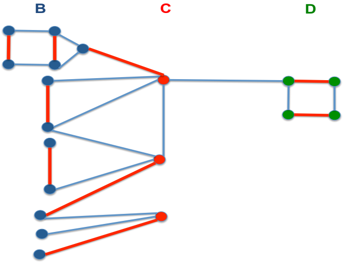


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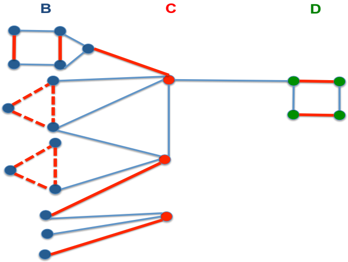


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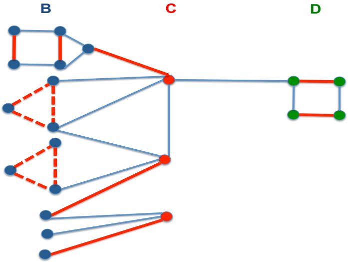


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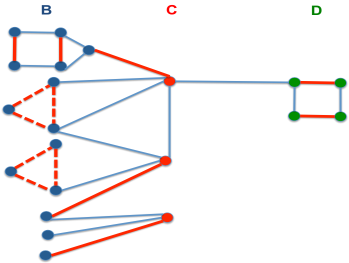


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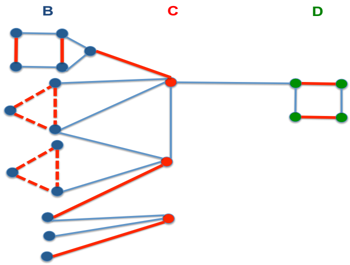


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How about approximation algorithms?

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