Templates for Minor-Closed Classes of Binary Matroids

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Some sets of columns are dependent, and some are independent.

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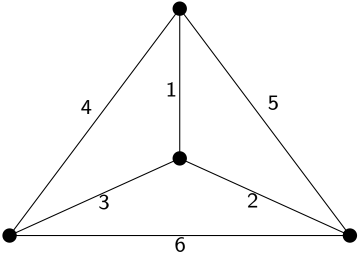

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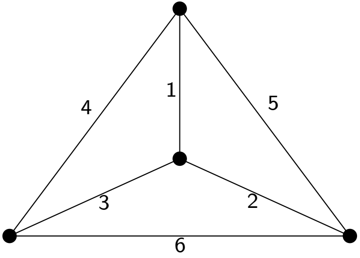

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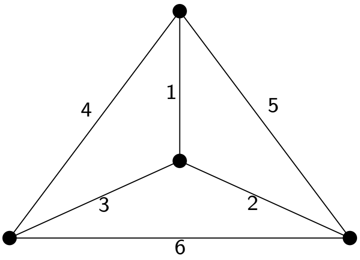

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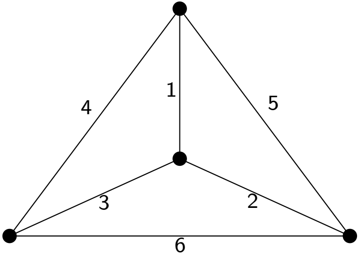

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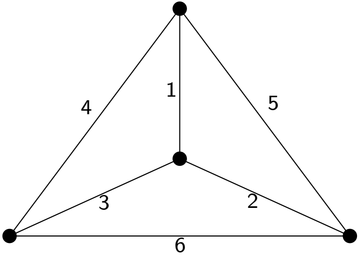

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All graphic matroids are binary.

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- Duals of graphic matroids are called *cographic* matroids.

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  - Templates help to specify what "close" means.
- Part of their profound structure theory of matroids representable over a finite field

Respecting a Template

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$\mathcal{M}(\Phi)$ is the set of matroids conforming to Φ.

Theorem (Geelen, Gerards, and Whittle 2015)

Let $\mathcal{M}$ be a proper minor-closed class of binary matroids. Then there exist $k, l \in \mathbb{Z}_+$ and frame templates $\Phi_1, \ldots, \Phi_s, \Psi_1, \ldots, \Psi_t$ such that

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- *$\mathcal{M}$ contains the duals of the matroids in each of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$, and*
- *if M is a simple vertically k-connected member of $\mathcal{M}$ with at least l elements, then either M is a member of at least one of the classes $\mathcal{M}(\Phi_1), \ldots, \mathcal{M}(\Phi_s)$, or M^* is a member of at least one of the classes $\mathcal{M}(\Psi_1), \ldots, \mathcal{M}(\Psi_t)$.*

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  - The relation $\preceq$ is a preorder on the set of frame templates.

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(vii) *There exist $k, l \in \mathbb{Z}_+$ such that no simple, vertically k-connected matroid with at least l elements either conforms or coconforms to Φ.*

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4. Otherwise, repeat Step (1).

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Conjecture (Seymour's 1-flowing Conjecture, 1981)

The set of excluded minors for the class of 1-flowing matroids consists of $U_{2,4}$, $AG(3,2)$, T_{11}, and T_{11}^*.

1-Flowing Matroids (cont.)

It can be shown that to each of Φ_{Y_0}, Φ_{Y_1}, Φ_C, Φ_X, and Φ_{CX} conforms a matroid with an $AG(3,2)$-minor.

1-Flowing Matroids (cont.)

It can be shown that to each of Φ_{Y_0}, Φ_{Y_1}, Φ_C, Φ_X, and Φ_{CX} conforms a matroid with an $AG(3,2)$-minor.
Thus, we have the following:

Theorem (G. and Van Zwam, 2017)

There exist $k, l \in \mathbb{Z}_+$ such that every simple, vertically k-connected, 1-flowing matroid with at least l elements is either graphic or cographic.

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Example: A 3-connected graph with at least 11 edges is planar if and only if it contains no $K_{3,3}$-minor.

$\mathcal{EX}(M_1, M_2, \dots)$: the class of binary matroids with no minor in the set $\{M_1, M_2, \dots\}$.

Even-Cycle and Even-Cut Matroids

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Theorem (G. and Van Zwam, submitted)

There exist $k, l \in \mathbb{Z}_+$ such that a vertically k-connected matroid with at least l elements is in $\mathcal{EX}(PG(3,2)\backslash e, M^*(K_6), L_{11})$ if and only if it is an even-cycle matroid.

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Theorem (G. and Van Zwam, submitted)

There exist $k, l \in \mathbb{Z}_+$ such that a cyclically k-connected matroid with at least l elements is in $\mathcal{EX}(M(K_6), H_{12}^*)$ if and only if it is an even-cut matroid.

Thank you!