Completing Some Partial Latin Squares

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Current Section

Partial latin squares

Definition 1

A partial latin square (PLS) of order n is an $n \times n$ array of n symbols in which each symbol occurs at most once in each row and column.

Partial latin squares

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Definition 2

A PLS of order n is called a latin square (LS) of order n if each cell is nonempty.

Partial latin squares

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Definition 2

A PLS of order n is called a latin square (LS) of order n if each cell is nonempty.

1		4		
2				3
	1		3	
		2		5
3				1

1	2	3	4	5
2	4	1	5	3
5	1	2	3	4
4	3	5	1	2
3	5	4	2	1

Completing PLS

Definition 3

A PLS P is called completable if there is a LS of the same order containing P.

Completing PLS

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1		3		
2				3
	1		3	
		5		2
3				1

1	2	3	4	5
2	4	1	5	3
5	1	2	3	4
4	3	5	1	2
3	5	4	2	1

Completing PLS

When can a PLS be completed?

Completing PLS

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1		3		
2				3
	2	4	3	5
		5		2
3				1

Completing PLS

When can a PLS be completed?

1		3		
2				3
	2	4	3	5
		5		2
3				1

- The problem of completing PLSs is NP-complete. (Colbourn, 1984)

Completing PLS

When can a PLS be completed?

1		3		
2				3
	2	4	3	5
		5		2
3				1

- The problem of completing PLSs is NP-complete. (Colbourn, 1984)
- A good characterization of completable partial latin square is unlikely.

Equivalent Objects

A PLS P of order n is a subset of $[n] \times [n] \times [n]$ in which $(r, c, s) \in P$ if and only if symbol s occurs in cell (r, c).

Equivalent Objects

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$(2, 1, 2), (4, 3, 5) \in P$

Equivalent Objects

A LS of order n is equivalent to a properly n-edge-colored $K_{n,n}$.

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$L =$

1	2	3
2	3	1
3	1	2

Equivalent Objects

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$L =$

1	2	3
2	3	1
3	1	2

Theorem 1

Let G be a bipartite graph with $\Delta(G) = m$. Then $\chi'(G) = m$.

Isotopisms and Congujates

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$.

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Let $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$.

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Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$.

Let $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$.

The PLS in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively is $\theta(P) \in \mathrm{PLS}(n)$.

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The mapping θ is called an isotopism, and P and $\theta(P)$ are said to be isotopic.

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$\theta(P) =$

3		1		
2				1
	3		1	
		5		2
1				3

Isotopisms and Congujates

Let $P \in \mathrm{PLS}(n)$ and S_n be the symmetric group acting on $[n]$.

Let $\theta = (\alpha, \beta, \gamma) \in S_n \times S_n \times S_n$.

The PLS in which the rows, columns, and symbols of P are permuted according to α, β, and γ respectively is $\theta(P) \in \mathrm{PLS}(n)$.

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$P =$

1		3		
2				3
	1		3	
		5		2
3				1

$\theta(P) =$

	1	3		
	2			3
1			3	
		5		2
	3			1

Isotopisms and Congujates

The PLS in which the coordinates of each triple of P are uniformly permuted is called a conjugate of P.

Isotopisms and Congujates

The PLS in which the coordinates of each triple of P are uniformly permuted is called a conjugate of P.

$P =$

1				
2				3
	1			
4				1

$P^{(rc)} =$

1	2			4
		1		
	3			1

Isotopisms and Congujates

The PLS in which the coordinates of each triple of P are uniformly permuted is called a conjugate of P.

$P =$

1				
2				3
	1			
4				1

$P^{(rs)} =$

1	3			5
2				
				2
5				

Isotopisms and Congujates

Theorem 2

A PLS P is completable if and only if an isotopism of P is completable.

Isotopisms and Congujates

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A PLS P is completable if and only if an isotopism of P is completable.

Theorem 3

A PLS P is completable if and only if a conjugate of P is completable.

Current Section

Hall's Theorem

Theorem 4 (Hall's Theorem, 1940)

Let $r, n \in \mathbb{Z}$ *such that* $r \leq n$. *Let* $P \in \mathrm{PLS}(n)$ *with* r *completed rows and* $n - r$ *empty rows. Then* P *can be completed to a LS of order* n.

Hall's Theorem

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Let $r, n \in \mathbb{Z}$ such that $r \leq n$. Let $P \in \mathrm{PLS}(n)$ with r completed rows and $n - r$ empty rows. Then P can be completed to a LS of order n.

Rows can be replaced with columns or symbols.

Hall's Theorem

1	2	3	4	5	6	7
2	6	1	7	3	4	5
5	1	7	3	4	2	6

Hall's Theorem

1	2	3				
2	6	1				
3	1	7				
4	5	6				
5	7	2				
6	4	5				
7	3	4				

Hall's Theorem

1	2	3				
2		1				3
3	1	2				
			1	2	3	
	3		2	1		
			3		1	2
				3	2	1

Ryser's Theorem

Theorem 5 (Ryser's Theorem, 1950)

Let $r, s, n \in \mathbb{Z}$ *such that* $r, s \leq n$. *Let* $P \in \mathrm{PLS}(n)$ *with a* $r \times s$ *block of symbols and empty cells elsewhere. Then* P *can be completed if and only if each symbol occurs* $r + s - n$ *times in* P.

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Let $r, s, n \in \mathbb{Z}$ such that $r, s \leq n$. Let $P \in \mathrm{PLS}(n)$ with a $r \times s$ block of symbols and empty cells elsewhere. Then P can be completed if and only if each symbol occurs $r + s - n$ times in P.

1	2	3				
2	4	5				
5	1	2				

1	2	3	7			
2	4	5	6			
5	1	2	4			
3	5	6	1			

1	2	3	5			
2	4	5	6			
5	1	2	4			
3	5	6	1			

Evans' Conjecture

Theorem 6

If $P \in \mathrm{PLS}(n)$ with at most $n-1$ non-empty cells, then P can be completed.

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1						
	4	5				
5						
			3			
				1		

1						
		5		4		
5						
			3			
	1					

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1					
		5		4	
5					
	1				

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7
2	7	5	1	4	6
5	4	6	2	7	1
6	5	1	7	2	4
7	6	2	4	1	5
4	1	7	6	5	2

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4	6	
5	4	6	2	7	1	
6	5	1	7	2	4	
7	6	2	4	1	5	
4	1	7	6	5	2	

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		2	4	7
7	6		4	1	5	2
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	1	5	2
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	2	5	1
4		7	6	5	2	1

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	
2	7	5	1	4		6
5	4	6	2		1	7
6	5	1		7	4	2
7	6		4	2	5	1
4		7	6	1	2	5

Evans' Conjecture

1						
	4	5				
5						
			3			
				1		

1	2	4	5	6	7	3
2	7	5	1	4	3	6
5	4	6	2	3	1	7
6	5	1	3	7	4	2
7	6	3	4	2	5	1
4	3	7	6	1	2	5

There are incompletable PLSs of order n with n non-empty cells.

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1				
2				
3				
4				
	5			

1	2	3	4	
				5

1				
	1			
		1		
			1	
				2

There are incompletable PLSs of order *n* with *n* non-empty cells.

1				
2				
3				
4				
	5			

1	2	3	4	
				5

1				
	1			
		1		
			1	
				2

1				
2				
3				
	4	5		

1	2	3		
			4	
			5	

1				
	1			
		1		
			2	
			3	

There are incompletable PLSs of order n with n non-empty cells.

1				
2				
3				
4				
	5			

1	2	3	4	
				5

1				
	1			
		1		
			1	
				2

1				
2				
3				
	4	5		

1	2	3		
			4	
			5	

1				
	1			
		1		
			2	
			3	

Let $B_{k,n} \in \mathrm{PLS}(n)$ with symbol 1 in the first k diagonal cells and symbols $2, 3, \ldots, n-k+1$ in the last $n-k$ cells of column $k+1$.

Theorem 7 (Andersen and Hilton, 1983)

Let $P \in \mathrm{PLS}(n)$ *with exactly* n *non-empty cells. Then* P *can be completed if and only if* P *is not a species of* $B_{k,n}$ *for each* $k < n$*.*

Current Section

Completed Rows and Columns

When can a PLS with exactly a rows and b columns be completed?

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

- Buchanan found all such PLSs for $a = b = 2$ in a 100 page dissertation (2007)

Completed Rows and Columns

When can a PLS with exactly *a* rows and *b* columns be completed?

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

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- Adam, Bryant, and Buchanan shortened Buchanan's case analysis to 25 pages (2008)

Completed Rows and Columns

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1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

- Buchanan found all such PLSs for $a = b = 2$ in a 100 page dissertation (2007)
- Adam, Bryant, and Buchanan shortened Buchanan's case analysis to 25 pages (2008)
- Kuhl and McGinn proved the same result and more (2017)

Completed Rows and Columns

$Y =$

1	2	3	4
3	4	2	1
2	3		
4	1		

$Z =$

1	2	3	4	5
3	1	2	5	4
2	3			
4	5			
5	4			

Completed Rows and Columns

$Y =$

1	2	3	4
3	4	2	1
2	3		
4	1		

$Z =$

1	2	3	4	5
3	1	2	5	4
2	3			
4	5			
5	4			

Let Γ denote the set of all isotopisms of Y and Z.

Theorem 8

Let $n \geq 2$ and $A \in \mathrm{PLS}(2, 2; n)$. The partial latin square A can be completed if and only if $A \notin \Gamma$.

Completed Rows and Columns

Suppose there is a filled cell of $A \in \mathrm{PLS}(2, 2; n)$ not in an intercalate.

Completed Rows and Columns

Suppose there is a filled cell of $A \in \mathrm{PLS}(2,2;n)$ not in an intercalate.

1	2	4	5	6	7	3
2	7	5	1	3	6	4
5	4					
6	5					
3	6					
4	1					
7	3					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3
2	7	1	3	6	5
7	3				
6	5				
3	6				
5	1				

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3
2	7	1	3	6	5
7	3	2	1	5	6
6	5	3	7	2	1
3	6	7	5	1	2
5	1	6	2	3	7

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6	5	
7	3	2	1	5	6	
6	5	3	7	2	1	
3	6	7	5	1	2	
5	1	6	2	3	7	

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6		5
7	3	2	1		6	5
6	5	3		2	1	7
3	6		5	1	2	7
5		6	2	3	7	1

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	
2	7	1	3	6		5
7	3	2	1		5	6
6	5	3		2	1	7
3	6		7	5	2	1
5		6	2	1	7	3

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3	2	1	4	5	6
6	5	3	4	2	1	7
3	6	4	7	5	2	1
5	4	6	2	1	7	3

Completed Rows and Columns

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3					
6	5					
3	6					
5	4					
4	1					

1	2	5	6	7	3	4
2	7	1	3	6	4	5
7	3	2	1	4	5	6
6	5	3	4	2	1	7
3	6	4	7	5	2	1
5	4	6	2	1	7	3
4	1	7	5	3	6	2

Completed Rows and Columns

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

1	2	4	3	7	6	4	5
3	4	2	1	6	7	5	4
4	3						
2	1						
7	6						
6	7						
4	5						
5	4						

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

1	2	3	4	6	5
2	3	1	5	4	6
3	1				
6	4				
5	6				
4	5				

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

1	2	3	4	6	5
2	3	1	5	4	6
3	1	4	6	5	2
6	4	5	1	2	3
5	6	2	3	1	4
4	5	6	2	3	1

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	
3	1	2	4	6	5	
1	2	3	5	4	6	
5	6	4	1	2	3	
2	5	6	3	1	4	
6	4	5	2	3	1	

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	
3	1	2	4	6		5
1	2	3	5		6	4
5	6	4		2	3	1
2	5		3	1	4	6
6		5	2	3	1	4

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	
3	1	2	4	6		5
1	2	3	5		6	4
5	6	4		2	3	1
2	5		3	1	4	6
6		5	2	4	1	3

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	1					
6	4					
4	6					
5	7					
7	5					

4	3	1	6	5	2	7
3	1	2	4	6	7	5
1	2	3	5	7	6	4
5	6	4	7	2	3	1
2	5	7	3	1	4	6
6	7	5	2	4	1	3
7	4	6	1	3	5	2

Theorem 9 (Kuhl and McGinn, 2017)

Let $A \in \mathrm{PLS}(2, b; n)$ and cells $[2] \times [b]$ consist only of symbols from $[b]$. If $n \geq 2b^2 - 2b + 5$ and $\sigma_A([n] \backslash [b])$ contains a cycle of length at least $\frac{n+3}{2}$, then A can be completed.

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Conjecture 1

Let $A \in \mathrm{PLS}(2, b; n)$. If $n \geq 2b + 2$, then A can be completed.

One Nonempty Row, Column, and Symbol

Theorem 10 (Kuhl and Schroeder, 2016)

Let $r, c, s \in \{1, 2, \ldots, n\}$ and let $P \in \mathrm{PLS}(n)$ in which each nonempty cell lies in row r, column c, or contains symbol s. If $n \notin \{3, 4, 5\}$ and row r, column c, and symbol s can be completed in P, then a completion of P exists.

One Nonempty Row, Column, and Symbol

Theorem 10 (Kuhl and Schroeder, 2016)

Let $r, c, s \in \{1, 2, \ldots, n\}$ and let $P \in \mathrm{PLS}(n)$ in which each nonempty cell lies in row r, column c, or contains symbol s. If $n \notin \{3, 4, 5\}$ and row r, column c, and symbol s can be completed in P, then a completion of P exists.

1	2	3
2	1	
3		1

1	3	4	2
2	1		
3		1	
4			1

1	3	2	4	5
2	1			
3		1		
4			1	
5				1

2	3		4	5
	1			
3		1		
4			1	
5				1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	7	2	6	3	4
2	1					
5		1				
3			1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5			1			
3		1				
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1				
3			1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1	4			
3		4	1			
4				1		
6					1	
7						1

One Nonempty Row, Column, and Symbol

1	5	2	6	7	3	4
2	1					
3		1				
4			1			
5				1		
6					1	
7						1

1	5	2	7	6	3	4
2	1					
5		1	4			
3		4	1			7
4				1	6	
6				4	1	
7			3			1

One Nonempty Row, Column, and Symbol

4	5	2	6	7	3	1
2					1	
3				1		
7			1			
5		1				
6	1					
1						