Arc Graphs and Posets

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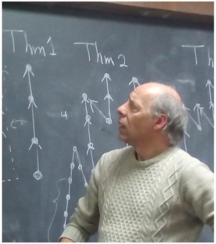

Claude Tardif

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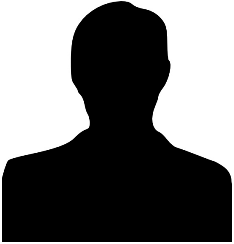


Imed Zaguia

What is an arc graph?

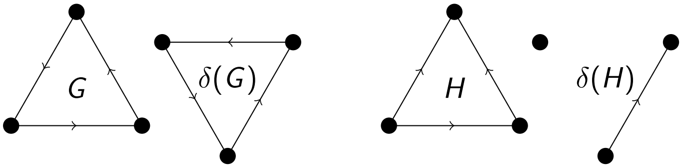
Arc graphs are line graphs of directed graphs.

Definition

The arc graph $\delta(G)$ of digraph G is the digraph with

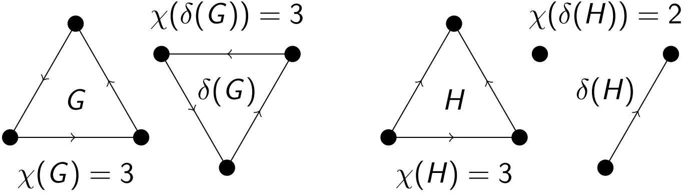
$$\begin{aligned} V(\delta(G)) &= A(G); \\ A(\delta(G)) &= \{uvw \mid uv, vw \in A(G)\}. \end{aligned}$$

Examples:



Chromatic number of a digraph

A proper coloring of a digraph is indifferent to arc direction.



Theorem (Entringer-Harner, 1972)

(i) *If* $\chi(\delta(G)) \leq n$, *then* $\chi(G) \leq 2^n$.

(ii) *If* $\chi(G) \leq \binom{n}{\lfloor n/2 \rfloor}$, *then* $\chi(\delta(G)) \leq n$.

Arc graph of symmetric graphs

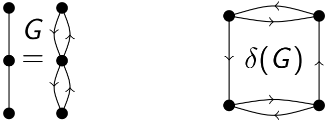
Something nice happens in the case of symmetric digraphs:

Theorem (Poljak-Rödl, 1981)

If G is an undirected graph, then

$$\chi(\delta(G)) = \min\left\{ n \mid \chi(G) \leq \binom{n}{\lfloor n/2 \rfloor} \right\}.$$

For undirected G, $\chi(\delta(G))$ depends only on $\chi(G)$ and not on the structure of G. What about $\chi(\delta(\delta(G)))$?



What about $\delta^\ell(G)$?

We show that $\chi(\delta^\ell(G))$ only depends on $\chi(G)$ for all ℓ when G is symmetric.

To do this, view δ as a digraph functor and define a "right adjoint" δ_R such that:

$$\big(\exists \text{ homom. } \delta(G) \to H\big) \iff \big(\exists \text{ homom. } G \to \delta_R(H)\big).$$

Once we define δ_R,

$$\begin{gathered} \delta^\ell(G) \text{ is } n\text{-colorable} \\ \Updownarrow \\ \text{there exists a homomorphism } \delta^\ell(G) \to K_n \\ \Updownarrow \\ \text{there exists a homomorphism } G \to \delta_R^\ell(K_n). \end{gathered}$$

Transitive digraphs

How can we deal with $\delta_R^\ell(K_n)$?

Posets!

K_n is the nondomination digraph $\mathcal{N}(\overline{K_n})$ of the n-element antichain.

Definition

The *nondomination digraph* $\mathcal{N}(P)$ of poset P has

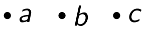
$$\begin{aligned} V(\mathcal{N}(G)) &= V(P); \\ A(\mathcal{N}(G)) &= \{uv \mid u \not\geq v \text{ in } P\}. \end{aligned}$$

Get down with the posets

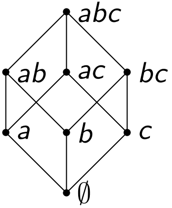
How do we deal with $\delta_R^\ell(\mathcal{N}(\overline{K_n}))$?

$\mathcal{I}(P)$ is the poset of ideals/downsets of P, ordered by inclusion. For example:

$\overline{K}_3$



$\mathcal{I}(\overline{K}_3)$



Lemma (RTWZ, 2016+)

For any poset P, there exist homomorphisms $\delta_R(\mathcal{N}(P)) \longleftrightarrow \mathcal{N}(\mathcal{I}(P))$.

Almost there...

$$\delta^\ell(G) \text{ is } n\text{-colorable}$$
$$\Updownarrow$$
$$\text{there exists a homomorphism } G \to \delta_R^\ell(K_n)$$
$$\Updownarrow$$
$$\text{there exists a homomorphism } G \to \mathcal{N}(\mathcal{I}^\ell(\overline{K_n})).$$

If digraph G is symmetric, we need only consider the symmetric edges of $\mathcal{N}(\mathcal{I}^\ell(\overline{K_c}))$.

Lemma

For poset P, there exist homomorphisms between ($\mathcal{N}(P)$ restricted to its symmetric edges) and (K_w with w the width of P).

The final stretch

$$\delta^\ell(G) \text{ is } n\text{-colorable}$$
$$\Updownarrow$$
$$\text{there exists a homomorphism } G \to \mathcal{N}(\mathcal{I}^\ell(\overline{K_n}))$$
$$\Updownarrow$$
$$\text{there exists a homomorphism } G \to K_w,$$
$$\text{with } w = \mathit{width}\big(\mathcal{I}^\ell(\overline{K_n})\big).$$

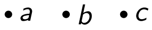
Theorem (RTWZ, 2016+)

For any undirected graph G and any integer $\ell \geq 1$,

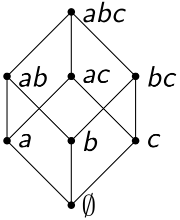
$$\chi\Big(\delta^\ell(G)\Big) = \min\Big\{n \mid \chi(G) \leq \mathit{width}\Big(\mathcal{I}^\ell(\overline{K_n})\Big)\Big\}.$$

Examples of $\mathcal{I}^{\ell}(\overline{K_n})$

$\overline{K}_3$



$\mathcal{I}(\overline{K}_3)$



$\mathcal{I}^2(\overline{K}_3)$