

A $(5, 5)$ -coloring of K_n with few colors

Alex Cameron

University of Illinois at Chicago
joint work with Emily Heath (UIUC).

May 20, 2017

Definition (Erdős; 1981)

A (p, q) -coloring of a graph is coloring of the edges such that every copy of K_p contains at least q distinct colors. Let $f(n, p, q)$ denote the minimum number of colors needed to (p, q) -color of the complete graph on n vertices, K_n .

- $1 \leq q \leq \binom{p}{2}$.
- $f(n, p, 1) = 1$.
- $f(n, p, \binom{p}{2}) = \binom{n}{2}$ for $p \geq 4$.
- Determining $f(n, p, 2)$ is equivalent to the classic Ramsey problem.
- $f(n, 3, 3) \approx n$.

Theorem (Erdős and Gyárfás; 1997)

The Local Lemma gives a general upper bound:

$$f(n, p, q) \leq cn^{\frac{p-2}{\binom{p}{2}-q+1}}.$$

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - p + 3$. Then $f(n, p, q) = \Theta(n)$ and $f(n, p, q - 1) \leq cn^{1 - \frac{1}{p-1}}$.

Theorem (Erdős and Gyárfás; 1997)

Fix p and let $q = \binom{p}{2} - \lfloor p \rfloor + 2$. Then $f(n, p, q) = \Theta(n^2)$ and $f(n, p, q - 1) \leq cn^{2 - \frac{4}{p}}$.

Erdős and Gyárfás gave a simple induction argument which demonstrates that

$$n^{\frac{1}{p-2}} - 1 \leq f(n, p, p),$$

the smallest value of q for which they could find a polynomial lower bound. They also considered several cases for small p :

- $n^{1/2} - 1 \leq f(n, 4, 4) \leq cn^{2/3}$ - one of the “most interesting” cases.
- $f(n, 4, 3) \leq cn^{1/2}$ - the “most annoying” case since unsure if it is even polynomial at all.

Theorem (Mubayi; 1998)

$$f(n, 4, 3) \leq e^{\sqrt{c \log n}(1+o(1))}.$$

Theorem (Conlon, Fox, Lee, and Sudakov; 2015)

$$f(n, p, p - 1) \leq 2^{16p(\log n)^{1-1/(p-2)} \log \log n}.$$

This shows that $q = p$ is the threshold at which $f(n, p, q)$ becomes polynomial in n .

Theorem (Mubayi; 2004)

$$f(n, 4, 4) \leq n^{1/2} e^{c\sqrt{\log n}}.$$

- This shows that $n^{1/2} \leq f(n, 4, 4) \leq n^{1/2+o(1)}$.
- Uses the product of two explicit colorings:
 - the construction showing that $f(n, 4, 3)$ is subpolynomial, and
 - an algebraic coloring which associates each vertex with a vector in \mathbb{F}_q^2 and uses a symmetric map $\mathbb{F}_q^2 \times \mathbb{F}_q^2 \rightarrow \mathbb{F}_q$ to color the edges.

Theorem (C. and Heath; 2017)

$$f(n, 5, 5) \leq n^{1/3} 2^{c\sqrt{\log n} \log \log n}.$$

- This shows that $n^{1/3} \leq f(n, 5, 5) \leq n^{1/3+o(1)}$.
- Uses the product of two explicit colorings:
 - the construction by Conlon, Fox, Lee, and Sudakov (CFLS) using $n^{o(1)}$ colors, and
 - an algebraic coloring which associates each vertex with a vector in \mathbb{F}_q^3 and uses a symmetric map $\mathbb{F}_q^3 \times \mathbb{F}_q^3 \rightarrow \mathbb{F}_q$ to color the edges.

The CLFS Coloring

Let $n = 2^{\beta^2}$ for some positive integer β . Associate each vertex of K_n with a unique binary string of length β^2 :

$$V = \{0, 1\}^{\beta^2}.$$

For any vertex $v \in V$, let $v^{(i)}$ denote the i th block of bits of length β in v so that

$$v = (v^{(1)}, \dots, v^{(\beta)})$$

where each $v^{(i)} \in \{0, 1\}^{\beta}$.

Between two vertices $x, y \in V$, the CFLS coloring is defined by

$$\varphi_1(x, y) = \left(\left(i, \{x^{(i)}, y^{(i)}\} \right), i_1, \dots, i_{\beta} \right)$$

where i is the first index for which $x^{(i)} \neq y^{(i)}$, and for each $k = 1, \dots, \beta$, $i_k = 0$ if $x^{(k)} = y^{(k)}$ and otherwise is the first index at which a bit of $x^{(k)}$ differs from the corresponding bit in $y^{(k)}$.

$$x = (0, 1, 1, 1, 0, 1, 0, 0, 1)$$

$$y = (0, 1, 1, 0, 0, 1, 0, 1, 1)$$

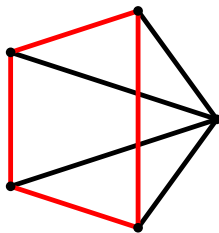
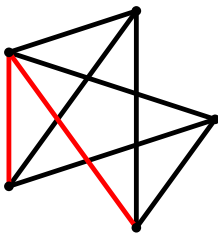
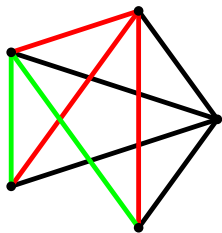
$$\varphi_1(x, y) = ((2, \{(0, 0, 1), (1, 0, 1)\}), 0, 1, 2)$$

Two easy and important facts about the CFLS coloring:

- Each color class is bipartite.
- Assume that all binary strings are ordered by the integers they represent so that the vertices are linearly ordered and also the individual blocks are ordered. Then, $a < b < c$ implies that $\varphi_1(a, b) \neq \varphi_1(b, c)$.

Problem Configurations

Three configurations not avoided by the modified CFLS coloring.



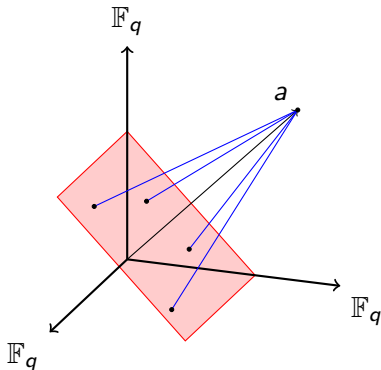
Let q be some odd prime power, and let \mathbb{F}_q^* denote the nonzero elements of the finite field with q elements. The vertices of our graph will be the three-dimensional vectors over this set,

$$V = \left(\mathbb{F}_q^*\right)^3.$$

The explicit definition of the coloring is a bit technical, but it is essentially the inner product of the two vectors with several modifications to take care of certain cases. I call it the Modified Dot Product (MDP) coloring.

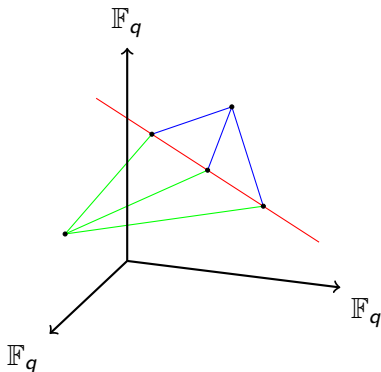
Monochromatic Neighborhoods

Given a vertex $a \in (\mathbb{F}_q^*)^3$ and a color $\alpha \in \mathbb{F}_q$, the monochromatic α -neighborhood of a is contained within an affine plane in \mathbb{F}_q^3 .



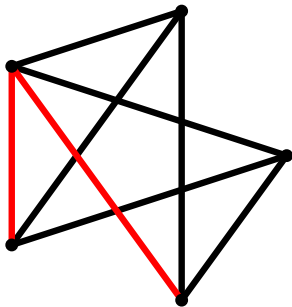
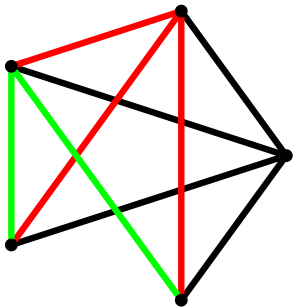
Intersection of Monochromatic Neighborhoods

“Most” of the time the intersection of two monochromatic neighborhoods defines a subset of an affine line.

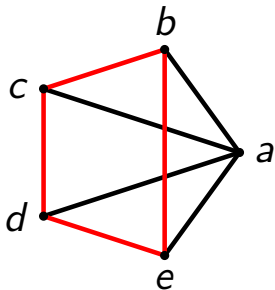


First Two Problems

Therefore, if we knew that the construction induced a *proper* edge coloring on every affine line, then any three vertices in the intersection of two monochromatic neighborhoods must span three distinct edge colors. This would get us close to eliminating the first two problem configurations.



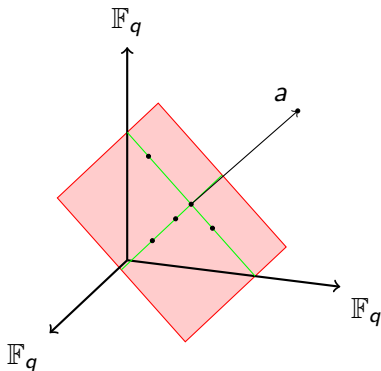
C_4 in a Monochromatic Neighborhood



This happens when the line defined by b and d is orthogonal to the line defined by c and e and these lines intersect at a scalar multiple of a .

Split the Coloring

This kind of thing happens a lot.



The solution is to partition the set of (non-isotropic) vectors in each linear plane into two sets so that no two in the same set are orthogonal. Then split each color into four colors based on these partitions.

- Is $f(n, p, p) \leq n^{1/(p-2)+o(1)}$ in general?
- Tighten other small cases:
 - $cn^{1/2} \leq f(n, 5, 6) \leq cn^{1/2+o(1)}$
 - $cn^{2/3} \leq f(n, 5, 7) \leq cn^{3/4}$
- Generalizations and related:
 - Geometric version: minimize number of distinct distances between n points in \mathbb{R}^d
 - $r(G, H, q)$ - minimum number of colors such that every copy of H in G receives at least q colors
 - (p, q_1, q_2) -colorings
 - bipartite version
 - hypergraph version