Games and Economic Behavior 76 (2012) 249–271

Contents lists available at SciVerse ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Political competition between differentiated candidates ☆

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ARTICLE INFO

Article history:
Received 2 December 2010
Available online 11 May 2012

JEL classification:
D72
D60

Keywords:
Multidimensional policy
Voting
Political competition

ABSTRACT

We introduce a framework of electoral competition in which voters have general preferences over candidates' immutable characteristics (such as gender, race or previously committed policy positions) as well as their policy positions, which are flexible. Candidates are uncertain about the distribution of voter preferences and choose policy positions to maximize their winning probability.

We characterize a property of voter utility functions ("uniform candidate ranking", UCR) that captures a form of separability between fixed characteristics and policy. When voters have UCR preferences, candidates' equilibrium policies converge in any strict equilibrium. In contrast, notions like competence or complementarity lead to non-UCR preferences and policy divergence. In particular, we introduce a new class of models that contains the probabilistic voting model as a special case and in which there is a unique equilibrium that generically features policy divergence.

1. Introduction

The political competition model introduced by Downs (1957) analyzes a setup in which two candidates each choose a platform from a one-dimensional set of feasible policies, such as the interval $[0, 1]$. All voters have single-peaked preferences over this policy space. If candidates are ex-ante identical and purely office-motivated, they propose identical policies to voters, namely the one that maximizes the utility of the median voter. The question we address in this paper is whether policy convergence is a robust feature of political competition if we admit a more general policy space and voter preferences, but otherwise keep Downs's assumption of two office-motivated candidates who compete under plurality rule.

To answer this question, we introduce a model where candidates have some unchangeable characteristics like their previous experience, gender or race. Candidates are flexible on policy issues, and they are willing to adopt positions to increase the probability of getting elected. Voters' preferences are defined over the candidates' vectors of characteristics and policies, and are completely general. The fundamental departure from the existing literature is that we do not require that preferences are separable across characteristics and issues. The distribution of voter preferences depends on a state variable that is unknown to candidates at the time they choose their positions.

Our first and straightforward result, Theorem 1, shows that differentiated fixed characteristics are necessary for (generic) equilibria with policy divergence to arise, even if we admit arbitrary voter preferences. Intuitively, without fixed characteristics, the candidates' payoffs from both choosing the same policy are equal to $1/2$. Since we have a constant sum game,

☆ We would like to thank three anonymous referees and an associate editor whose comments significantly improved this paper. We are also grateful to seminar audiences in UC Berkeley, Clemson, Harvard-MIT and University of Southern California, in particular Ernesto Dal Bo, Matthias Messner, Maggie Penn, Micah Pollak, Ken Shepsle and Jim Snyder for helpful comments.

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0899-8256/$ – see front matter
http://dx.doi.org/10.1016/j.geb.2012.05.003

the winning probabilities in *any* pure strategy equilibrium must be $1/2$ for each candidate and, generically, this can only be the case if the candidates choose the same policy.

This argument, however, breaks down if candidates have differentiated fixed characteristics. In that case, because candidates cannot perfectly copy their opponent, they may choose different policies and have unequal winning probabilities in equilibrium: Even if a candidate chooses the same policy platform as his opponent, the existence of differentiated fixed characteristics implies that many or all voters can still have strict preferences for one of the candidates. Hence, there is no guarantee that imitating the opponent will increase the winning probability of a candidate.

This insight raises the question of whether there is a class of voter preferences for which equilibrium policy convergence is still guaranteed, even if candidates have differentiated fixed characteristics. We find such a general property of voter preferences that we call *uniform candidate ranking* (UCR). UCR does not impose any restrictions on voter preferences if candidates choose different policies, but if the two candidates choose the same policy platform, a UCR voter always prefers the same candidate. That is, suppose that, due to the difference in fixed characteristics, a voter prefers Candidate 0 to Candidate 1 if both propose policy a; then a UCR voter also prefers Candidate 0 to Candidate 1 if both propose policy a'. Since every voter votes the same, whether both candidates choose a or both choose a', the UCR assumption implies that the payoffs for the candidates when they choose the same policy must always be the same.[1] Using this observation and the fact that the game is a zero sum game, Theorem 3 shows that there is policy convergence in any strict Nash equilibrium of a voting game with UCR preferences and ex-ante non-identical candidates.

Are UCR preferences a *necessary* condition for equilibrium policy convergence? Absent additional conditions, we cannot expect any assumption on individual preferences to be necessary. For example, if citizens with non-UCR preferences are never pivotal, or if UCR is violated for some policies that are undesirable for sufficiently many voters, then the violation of UCR would not matter for equilibrium convergence. However, we show that UCR preferences are "close to necessary", in the following sense: If we endow just one individual with non-UCR preferences, then there always exists a voting game (even in large electorates) where all other voters have UCR preferences and the unique strict Nash equilibrium has policy divergence (Theorem 4). This type of "necessity" is analogous to that of single-peaked preferences for the existence of a Condorcet winner: Single-peakedness of all voters' preferences is not necessary for a Condorcet winner to exist, but the existence of a single voter whose preferences are not single-peaked can lead to the non-existence of a Condorcet winner. As an alternative approach to show that UCR is "close to necessary" for policy convergence, Section 5 shows a class of models in which equilibrium policies converge if and only if preferences are UCR.

Most voter preferences found in the literature — such as the one-dimensional Downsian model, the Downsian model with uncertainty about the median, the Downsian model with valence, or the probabilistic voting model — are additively separable between fixed characteristics and flexible issues and can easily be shown to satisfy UCR (in fact, Theorem 2 shows that the class of UCR preferences is more general than the class of additively separable preferences). However, there are also natural circumstances in which voters have non-UCR preferences, and where policies diverge in a robust pure strategy equilibrium.

In Section 5, we present such a class of models that captures the notion of complementarity by generalizing the classical probabilistic voting model (PVM). In the PVM, groups are identified as voters who have the same "economic" preferences (i.e., preferences over policies chosen by the candidates), but within a group, voters may differ with respect to "ideology". Most papers in the probabilistic voting literature operationalize the notion of ideology through an additive ideology shock to the economic preferences, but one way to think about ideology is that it captures utility derived from the candidates' positions on a second policy dimension, orthogonal to fiscal policy, in which candidates cannot make credible commitments, but set an optimal policy after the election according to their preferred position.

We explicitly model the relevant policy space as two-dimensional: In one dimension, candidates are exogenously fixed while they can choose their policy position in the other dimension. Intuitively, indifference curves that are circles capture preferences where the ideal policy a is independent of the fixed characteristic c. In this case, the model has a unique equilibrium with convergence that corresponds to the equilibrium of the standard PVM. In contrast, consider, for example, elliptical indifference curves for which the major axis is the 45 degree line. This captures a situation where the fixed characteristic and the policy are complements in the sense that, the higher is the candidate's fixed characteristic, the higher is the voter's ideal policy a. For example, suppose that two presidential candidates differ in their posture towards international security cooperation (e.g., how willing they are to work within the framework of international organization, or also by how much cooperation/opposition these candidates would get from international actors). Candidates are fixed in this dimension, but they can choose the size of their proposed military spending. In this context, it is not implausible that a voter's ideal defense budget depends on the candidate's identity, i.e., his fixed characteristics.

With elliptical preferences, the model still has a unique equilibrium, but one that features policy divergence. Specifically, the candidate with a higher fixed characteristic chooses a higher position on the flexible policy dimension than his opponent. From a technical point of view, the model shows the surprising usefulness of Theorem 3 in a setting with non-UCR preferences. Specifically, we show that the policy space can be transformed in a way that voter preferences are UCR in the transformed policy space. We then apply Theorem 3 to show that the equilibrium in the transformed space is unique and

[1] Of course, in contrast to the case without fixed characteristics (or with identical fixed characteristics), the winning probabilities on the main diagonal do not have to be $1/2$.

features convergence. Re-transformation of the policy space then shows that the equilibrium in the actual policy space is still unique, but features divergence.

This new class of non-UCR models provides us with a tractable model in which purely office-motivated candidates choose divergent policy platforms. This result is due to complementarity between fixed characteristics and policies for voters, and contrasts with the policy convergence result for office motivated candidates in the standard model. One of the most popular models used to explain policy divergence within the standard spatial framework assumes that candidates are policy-motivated, i.e., candidates are willing to lower their chance of winning in the election in exchange for being able to implement a particular policy in case they win. Thus, the reader may ask whether we need an explanation other than policy motivation for policy divergence, and whether our model is empirically distinguishable from the model with policy-motivation.

Concerning the first question, we do not see our assumption of "office-motivation" as diametrically opposed to policy-motivation. It is quite plausible that candidates are policy-motivated in some issues, but these issues can be captured as "fixed positions" in our framework. Candidates use the positions on the *remaining* issues as tools to get elected — either because they care about the material aspects of the office (classical office-motivation), or because they care primarily about the implementation of their core convictions.[2] Explaining policy divergence *on flexible issues* in this framework is useful, because by focusing on the standard model of policy motivation, we may miss other interesting and relevant reasons why divergence arises in practice. In particular, in our model, divergence may be a strategy that maximizes a candidate's probability of winning, and thus would not have to be interpreted as an indication that the candidate is policy-motivated.

Related to the second question, the candidates' incentives that generate policy divergence differ between our model and the standard model with policy-motivated candidates. These different incentives can be used to generate testable predictions that allow us to empirically discriminate between the two models. In the standard spatial model, there are costs and benefits of policy divergence. By choosing a platform farther away from his opponent's, a candidate trades off an increased utility from policy if he wins against a lower chance of winning. In our model, candidates are assumed to maximize the probability of winning, and in some situations, this will induce them to choose positions that diverge from their opponent's equilibrium position. Thus, changes in the environment that affect the costs and benefits (e.g., an increase in the wage of the office-holder) should affect policy positions in the Downsian model, but not in ours. Similarly, the cost of policy divergence (in terms of reduction of the winning probability) is affected by the quality of information about the median voter's preferred position. Better and more easily available opinion polls should translate into smaller policy divergence in the standard model. In contrast, the extent of equilibrium divergence in Section 5 is independent of the uncertainty about voter preferences, and thus of the availability and quality of opinion polls.

2. Previous literature

The platform choice of candidates for political office is one of the major areas of interest in formal models of politics. There is a huge literature on the topic of policy convergence or divergence in one-dimensional models (or models with one policy dimension and one valence dimension). For excellent reviews of this area, see, e.g., Osborne (1995) and Grofman (2004).

There is a large literature that tries to explain, within the Downsian model, the empirical observation that candidates often propose considerably divergent policies. Candidates may diverge even though this decreases their winning probability, because they care about the implemented policy (see, e.g., Wittman, 1983; Calvert, 1985; Roemer, 1994; Martinelli, 2001; Gul and Pesendorfer, 2009). In contrast, in our model, divergence may increase a candidate's probability of winning.

Some models obtain policy divergence with office-motivated candidates in a one-dimensional setting with incomplete information among voters about candidate characteristics (e.g. Callander, 2008) or among candidates about the position of the median voter (Castanheira, 2003; Bernhardt et al., 2006). Another branch of literature on divergence with office motivation, which is less directly related to our paper, explains policy divergence as entry deterrence by two dominant parties (e.g., Palfrey, 1984; Callander, 2005).

Both the literature on candidates with valence (e.g. Ansolabehere and Snyder, 2000; Groseclose, 2001, 2007) and the probabilistic voting literature (e.g., Hinich, 1978; Lindbeck and Weibull, 1987, 1993; Coughlin, 1992; Dixit and Londregan, 1995; Banks and Duggan, 2005) share with our paper the feature that voters care both about candidates' unchangeable characteristics and their flexible policy positions. However, voter preferences in all these papers satisfy our UCR-property and thus, by Theorem 3, any pure strategy equilibrium in these models features convergence.

Krasa and Polborn (2010a) analyze a model with office-motivated candidates in which both fixed characteristics and flexible positions are binary and voters have an additively separable utility function. The main focus of Krasa and Polborn (2010a) is to characterize voter preference distributions for which candidates have "majority-efficient" positions, and under which conditions candidates choose majority-efficient positions in settings where those exist (a position on flexible issues is majority-efficient if there is no other position that a majority of voters would prefer from that candidate). Since additive

[2] By interpreting (some) fixed characteristics as already committed policy positions based on candidates' "core convictions" while preserving an instrumental interpretation of policy choices on other issues, our model also provides a middle ground between Downsian models, in which candidates are free to choose any position, and the citizen candidate model in which no commitment is possible.

voter preferences satisfy our UCR condition, any equilibrium "divergence" in Krasa and Polborn (2010a) is in mixed strategies only. In contrast, in Section 5 of the present paper, we show that divergence can arise in a strict pure strategy Nash equilibrium when voter preferences are of the non-UCR type.

There are a few dispersed papers in the literature in which voters are endowed with non-UCR preferences and in which a pure strategy equilibrium thus (can) feature policy-divergence. For example, Adams and Merrill (2003) analyze a model in which voters have, in addition to preferences over policy positions from the $[0, 1]$ interval, "non-policy preferences" over the two candidates, which corresponds to different fixed positions in our setting. They assume that voters may abstain due to being almost indifferent between candidates, or due to "alienation" (if their preferred candidate does not provide them with sufficient utility). While there is still policy convergence in this model if voters only abstain from indifference (see also Erikson and Romero, 1990), they show that abstention from alienation may provide an incentive for strong divergence. We show that abstention due to alienation leads to non-UCR preferences, which is the fundamental reason for divergence in Adams and Merrill (2003). Similarly, in a variation of their basic probabilistic voting model of redistribution between different economic groups, Dixit and Londregan (1996) show that, if candidates differ in how well they can transfer resources to different interest groups, then they usually propose different transfers.

Finally, Soubeyran (2009), Krasa and Polborn (2010b, 2011) and Jensen (2009) analyze settings in which candidates differ in their ability to implement certain policies. In these settings, competence differentials give rise to non-UCR preferences in a natural way. In all of these papers, the focus is on the particular application, while our main interest here is to understand which general properties of voter utility functions drive policy convergence or divergence results.

3. The model

Two candidates, $j = 0, 1$, compete in an election. Candidates are office-motivated and receive utility 1 if elected, and utility 0 otherwise, independent of the implemented policy. Candidate j has fixed characteristics $c_j \in C$, which we also call his *type*. If elected, Candidate j implements a policy position $a_j \in A$.

Uncertainty about voter preferences is described by a probability space $(\Omega, \mathfrak{O}, \mu)$: A state $\omega \in \Omega$ determines voters' preferences over $C \times A$, and μ is the probability distribution of these "preference shocks", while $\mathfrak{O}$ is the set of measurable events. In particular, let P_r be the set of preferences on $C \times A$. Then the preferences of voter $\ell \in \mathfrak{L} = \{1, \ldots, L\}$ in state $\omega \in \Omega$ are $\succcurlyeq^\ell_\omega \in P_r$.[3]

The timing of the game is as follows:

Stage 1 Candidates $j = 0, 1$ simultaneously announce policies $a_j \in A$. A mixed strategy by Candidate j consists of a probability distribution σ_j over A.

Stage 2 State $\omega \in \Omega$ is realized and each citizen votes for his preferred candidate, or abstains when he is indifferent.[4]

We consider two different objectives for the candidates, maximizing the probability of winning, and maximizing the expected vote share.[5]

Objective 1: Probability of winning maximization.

Candidate j wins the election if he receives more votes than his opponent. In case of a tie between the candidates, each wins with probability 1/2. Let $W^j(\omega, a_0, a_1)$ denote Candidate j's winning probability in state ω, given policies a_0 and a_1. Formally, $W^0(\omega, a_0, a_1) = \xi(\nu(\omega, a_0, a_1))$, where $\xi(x < 0) = 0$, $\xi(0) = 1/2$ and $\xi(x > 0) = 1$; and $\nu(\omega, a_0, a_1) = \#\{\ell | (c_0, a_0) \succcurlyeq^\ell_\omega (c_1, a_1)\} - \#\{\ell | (c_1, a_1) \succcurlyeq^\ell_\omega (c_0, a_0)\}$. Candidate 1's winning probability is given by $W^1(\omega, a_0, a_1) = 1 - W^0(\omega, a_0, a_1)$.

Objective 2: Vote share maximization.

Candidate 0's vote share in state ω is given by

$$V^0(\omega, a_0, a_1) = \frac{\#\{\ell | (c_0, a_0) \succ^\ell_\omega (c_1, a_1)\}}{\#\{\ell | (c_0, a_0) \not\sim^\ell_\omega (c_1, a_1)\}},$$

and Candidate 1's vote share is $V^1(\omega, a_0, a_1) = 1 - V^0(\omega, a_0, a_1)$.

[3] More formally, let $\mathfrak{P}_r$ be a σ-algebra of measurable subsets of P_r then voter ℓ's random preferences are given by a measurable function $t_\ell : \Omega \to P_r$. For example, if C and A are finite then P_r is finite. In this case, $\mathfrak{P}_r$ is the set of all subsets of P_r, and measurability means that the set of all states ω that are mapped into one particular preference ordering is measurable.

[4] If a voter has a strict preference, then it is a weakly dominant strategy to vote for the preferred candidate. If a voter is indifferent, he could in principle vote for any candidate or abstain. We assume that he abstains, which is quite natural (e.g., in the presence of even very small voting costs), and also allows us to easily model a random number of voters $L(\omega) \leqslant L$ by simply modeling $L - L(\omega)$ voters as indifferent between all policies, so that they will abstain no matter what policies the candidates choose.

[5] Note that it is interesting to investigate both objectives, since they lead in general to different equilibria (see Patty, 2005, 2007).

Definition 1.

1. Consider the game where candidates maximize their respective winning probability;
   (a) (a_0, a_1) is a *pure strategy Nash equilibrium* if and only if

$$\int W^0(\omega, a_0, a_1)\,d\mu(\omega) \geqslant \int W^0\big(\omega, a_0', a_1\big)\,d\mu(\omega) \quad \text{and}$$
$$\int W^1(\omega, a_0, a_1)\,d\mu(\omega) \geqslant \int W^1\big(\omega, a_0, a_1'\big)\,d\mu(\omega);$$

   for all $a_0', a_1' \in A$.
   (b) (a_0, a_1) is *a strict Nash equilibrium* if and only if the above inequalities are strict for all $a_0' \neq a_0$, and $a_1' \neq a_1$.
   (c) A pair of probability distributions (ρ_0, ρ_1) on A is a *mixed strategy Nash equilibrium* if and only if $a_0 \in \arg\max \int W^0(\omega, a_0, a_1)\,d\mu(\omega)\,d\rho_1(a_1)$ for all a_0 in the support of ρ_0, and $a_1 \in \arg\max \int W^1(\omega, a_0, a_1)\,d\mu(\omega)\,d\rho_0(a_0)$ for all a_1 in the support of ρ_1.
2. To get the corresponding definitions for the game with vote share maximization, replace W^0 by V^0 and W^1 by V^1.

4. Convergence and divergence of equilibrium policies

4.1. A general convergence result without fixed characteristics

Our first result shows that, for arbitrary voter preferences, if candidates' fixed characteristics coincide, then any generic pure strategy equilibrium displays policy convergence. Note that Theorem 1 is a characterization result and does not provide conditions under which a strict Nash equilibrium exists. Indeed, since our framework is very general, necessary and sufficient conditions for equilibrium existence are hard to obtain. Nevertheless, we know that Theorem 1 is not vacuous as there are classes of voter preferences, such as the Downsian model or the probabilistic voting model, in which a strict equilibrium is known to exist. The main usefulness of Theorem 1 is therefore that it tells modelers that, as long as candidates are identical, *no* utility functions for voters will be able to generate equilibrium divergence.

Theorem 1. *Suppose that $c_0 = c_1$. Then the following holds in the game with winning probability maximization and the game with vote share maximization*:

1. *If there exists a pure strategy Nash equilibrium (a_0, a_1) with $a_0 \neq a_1$, then (a_0, a_0) and (a_1, a_1) are also pure strategy Nash equilibria.*
2. *If there exists a strict Nash equilibrium (a_0, a_1) then $a_0 = a_1$ and this strict Nash equilibrium is the unique Nash equilibrium (pure or mixed).*

Divergent pure strategy equilibria cannot be unique, as long as candidates' fixed characteristics do not differ: Whenever they exist, there is also an equilibrium with policy convergence; moreover, any policy divergence is weak in the sense that candidates do not strictly prefer the particular platform they choose. Thus, our result generalizes the convergence results familiar from the Downsian model to a setup with multiple issues and uncertainty about preferences. In the Downsian model under certainty both candidates choose the policy that is most preferred by the median voter. If the position of the median voter is uncertain, then candidates converge on the "median median", that is, there is no other position that would make a majority better off in a majority of states. The intuition of the median voter theorem continues to hold for general preferences: In an equilibrium, no other position can make a majority of voters better off in a majority of states. The reason is that, if such a policy position existed, then either candidate could deviate to it, thereby increasing his winning probability to more than 1/2.

Theorem 1 is related to Theorem 7.1 in Austen-Smith and Banks (2005). In a setting with certainty about the preference distribution of voters, they show that a pair of platforms (a_0, a_1) is an equilibrium if and only if a_0 and a_1 are both policies that cannot be blocked by a decisive coalition (i.e., in the case of plurality rule, that are Condorcet winners). In many frameworks, there is (at most) one Condorcet winner, in which case convergence arises trivially. However, even if this is not the case, Theorem 1 shows that divergent equilibria can neither be strict nor unique.

Finally, it is quite clear that Theorem 1 cannot hold if there are more than two identical candidates. To see this, suppose that there are three candidates, and there is just one binary issue and two states of the world; in state 0, which obtains with probability 0.6, all voters prefer position 0, and in state 1, which obtains with probability 0.4, all voters prefer position 1. In this case, it is clearly a strict equilibrium that two candidates choose position 0 and the third one chooses position 1, leading to winning probabilities of 0.3 for each of the two candidates who share position 0 (assuming that voters randomize between them in state 0), and 0.4 for the candidate in position 1 who wins in state 1. It is also obvious that $(0, 0, 0)$ is not an equilibrium, because (for example) the third candidate could deviate to 1 and increase his winning probability from 1/3 in $(0, 0, 0)$ to 0.4 in $(0, 0, 1)$.

4.2. UCR preferences

We now turn to the more relevant case that candidates' fixed characteristics differ, and analyze under which conditions there is policy convergence in those issues that candidates are free to choose. In this section, we identify a condition on voter preferences called uniform candidate ranking (UCR). In Section 4.3, we show that UCR preferences are sufficient for equilibrium policies to (generically) converge, and that they are, in a certain sense, also necessary for convergence results.

We start with the definition of UCR preferences. Suppose that both candidates choose the same policy $a \in A$. We say that a voter has *uniform candidate ranking (UCR)* preferences if his preferences for the candidates are independent of a. For example, suppose that $C = A = \{0,1\}$. Preferences are therefore defined on $\{0,1\} \times \{0,1\}$, where the first coordinate is the candidate's fixed characteristic and the second one the policy issue. A UCR voter prefers $(0,0)$ to $(1,0)$ if and only if he also prefers $(0,1)$ to $(1,1)$.

Definition 2. Preferences $\succcurlyeq$ on $C \times A$ allow for a *uniform candidate ranking (UCR)* if, for all $c_0, c_1 \in C$ and all $a, a' \in A$,

$$(c_0, a) \succcurlyeq (c_1, a) \quad \text{if and only if} \quad \bigl(c_0, a'\bigr) \succcurlyeq \bigl(c_1, a'\bigr). \tag{1}$$

Models in which candidates have no fixed characteristics (e.g., the standard one-dimensional Downsian model) automatically satisfy Definition 2. Also, a model with a one-dimensional policy space and random candidate valences satisfies UCR, as does a model with uncertainty about the preferred position of the median voter (as well as valence). Likewise, voter preferences in the probabilistic voting model (see, e.g., Lindbeck and Weibull, 1987, 1993; Coughlin, 1992) satisfy UCR.

For example, consider a model with stochastic valence: In state $\omega = (\omega_0, \omega_1)$, voter θ's utility from Candidate 0 is given by $\omega_0 - (a_0 - \theta)^2$, while his utility from Candidate 1 is given by $\omega_1 - (a_1 - \theta)^2$. Clearly, when $a_0 = a_1$, the voter strictly prefers Candidate 0 if and only if $\omega_0 > \omega_1$. Since this preference is independent of the particular policy $a_0 = a_1$, UCR is satisfied.

Note that Definition 2 refers to pairwise comparisons of candidates (consisting of fixed and flexible policies). Thus, whether UCR holds is a property of utility functions and therefore independent of the actual number of candidates. While we focus on settings in which two candidates compete against each other, Definition 2 would remain unchanged if there are more than two candidates. Of course, if the two candidates c_0, c_1 are already fixed, we can effectively restrict the set C to contain exactly these two values, which makes it easier for preferences to satisfy UCR. That is, because (1) has to hold for all pairs of fixed characteristics in C, there are preferences that would fail UCR on a very general set of candidate-fixed characteristics C, but that satisfy UCR for a given specific pair of candidates, $C = \{c_0, c_1\}$.

While UCR preferences are prevalent in the literature, there are natural circumstances in which preferences violate UCR. For example, suppose that a candidate's fixed characteristics capture his competence in implementing different policies. Specifically, suppose that the fixed characteristic is whether or not a candidate has served in the military, while the policy issue is whether or not to go to war with some other country. It is conceivable that a voter considers the candidate who has served in the military as a better leader for the country during a war, while preferring his opponent with a civilian background if there is peace. Formally, such a voter could have the preference $(1,1) \succ (0,0) \succ (1,0) \succ (0,1)$, that is, prefers most to go to war with a leader with military experience, while the second best option is not to go to war and have a leader with civilian background, which again is better than both "mixed" policy vectors. These preferences violate UCR, because the voter's preferred candidate changes from the situation that both propose to go to war to another one in which both propose peace.

We now characterize the set of utility functions that represent UCR preferences.

Theorem 2. *Let A and C be separable metric spaces, and let C be compact. Then the following statements are equivalent*:

1. *Rational (i.e., complete and transitive) and continuous*[6] *preferences $\succcurlyeq$ on $C \times A$ satisfy UCR.*
2. *The preferences $\succcurlyeq$ can be described by a continuous utility function $u(c,a) = g(f(c),a)$ where $f: C \to Y \subset \mathbb{R}$ is continuous, and $g: Y \times A \to \mathbb{R}$ is continuous and strictly monotone in $y \in Y$.*[7]

We can interpret $f(c)$ as the voter's ranking of the candidates' fixed characteristics — a higher value of $f(c)$ indicates that the voter ranks the candidate higher, since g is strictly monotone in $f(c)$. Thus, a voter's preferences satisfy UCR if and only if there is such a ranking that is independent of policy a.

If the utility function is additively separable across A and C, i.e., $u(c,a) = u_C(c) + u_A(a)$, then Theorem 2 immediately implies that preferences satisfy UCR. Suppose, for example, that $C \subset \mathbb{R}$ and that $A = \prod_{i=1}^{I} A_i$ (i.e., there are I different issues). Thus, a candidate's policy can be written as $a = (a_1, \ldots, a_I)$, and the "weighted issue preferences" of Krasa and Polborn (2010a), can be represented by the additively separable utility function

[6] Note that continuity is automatically satisfied if A and C are finite.

[7] Note that Y inherits its topology as well as its ordering from the reals.

$$u(a,c) = -\lambda_C|c-\theta_C|^q - \sum_{i=1}^{I}\lambda_i|a_i-\theta_i|^q, \tag{2}$$

where $q > 0$. Parameters θ and λ can be interpreted as ideal positions and weights that measure the relative importance of the fixed and selectable issues, respectively.[8] Another class of preferences with additively separable utility function are those where indifference curves are circles around an ideal point θ. While additive separability guarantees that UCR holds, the following example shows that it is not a necessary condition.

Example 1. Let $c_0 = 0$, $c_1 = 1$, and assume that there is only one binary policy issue, i.e., $A = \{0,1\}$. The voter's preference is $(0,0) \succ (0,1) \succ (1,1) \succ (1,0)$. Clearly, UCR is satisfied, as Candidate 0 is always preferred to Candidate 1. However, these preferences cannot be represented by an additively separable utility function $u_C(c) + u_A(a)$ because $(0,0) \succ (0,1)$ would imply $u_A(0) > u_A(1)$, while $(1,1) \succ (1,0)$ would imply $u_A(1) > u_A(0)$, a contradiction.

4.3. Convergence and divergence

The following Theorem 3 again considers the topic of convergence, but in contrast to Theorem 1, it allows for candidates' fixed positions to differ and focuses on the case that all voters have UCR preferences for a.e. realization of $\omega \in \Omega$. Under these conditions, there is policy convergence in all strict Nash equilibria. Moreover, if a strict Nash equilibrium exists, then it is unique.

Theorem 3. *Suppose that all voters have UCR preferences for a.e. realization of $\omega \in \Omega$ (c_0 and c_1 are arbitrary, in contrast to Theorem 1). Then the following holds in the game with winning probability maximization and the game with vote share maximization*:

1. *There is policy convergence in any strict Nash equilibrium (a_0, a_1), i.e. $a_0 = a_1$.*
2. *If there exists a strict Nash equilibrium then it is the unique Nash equilibrium (pure or mixed).*

It is useful to discuss here the intuition for how the UCR assumption shapes Theorem 3. For comparison, remember that, in the case that candidates do not differ in fixed characteristics, the fact that a candidate can always copy his opponent and thereby secure a winning probability of 1/2 implies that strict equilibria cannot be off the diagonal. In contrast, with different fixed characteristics, UCR preferences allow for potentially asymmetric payoffs for the two candidates. However, the key feature of UCR preferences is that it is still true that each candidate can always secure a particular set of supporters by copying his opponent. This feature again implies that strict equilibria cannot be off the diagonal — reverting to the diagonal by copying the opponent is either attractive for Candidate 0 or for Candidate 1.

More formally, suppose both candidates choose the same policy a. Since voters have UCR preferences, the winning probabilities do not change if both candidates switch to a'. This means that the entries on the diagonal of the payoff matrix (i.e., where $a_0 = a_1$) are identical, though not necessarily equal to 1/2. Suppose, by way of contradiction, that there is a strict Nash equilibrium (a_0, a_1), with $a_0 \neq a_1$. This would require that Candidate 0 strictly prefers his payoff in (a_0, a_1) to his payoff in (a_1, a_1), i.e., the payoff that he could obtain by deviating to a_1. Similarly, Candidate 1 strictly prefers his payoff in (a_0, a_1) to his payoff in (a_0, a_0). However, since the candidates play a constant sum game and the payoffs in (a_0, a_0) and (a_1, a_1) are equal because of UCR, we get a contradiction.

The proof of Theorem 3 relies on the fact that candidates play a constant-sum game in our model, whether they care about their probability of winning or their vote share. As Zakharov (2012) has shown, if candidates for political office are assumed not to be in pure conflict (i.e., their utilities as a function of the votes they receive do not sum up to a constant), then policy divergence may arise even if voters have UCR preferences.

One of the very few models with an equilibrium in which office-motivated candidates choose divergent platforms is Adams and Merrill (2003). Theorem 3 indicates that this must be due to non-UCR preferences in their model. Voters in their model have additively separable preferences that incorporate both a (continuous) policy issue and partisan preferences (akin to "fixed characteristics" in our terminology). Specifically, consider the following example.

Example 2. There is one fixed characteristic, which Adams and Merrill (2003) refer to as partisanship, and a one-dimensional policy variable in [0, 1]. A citizen's type is of the form (P, θ), where $P \in \{D, R\}$ denotes the partisan preference, and θ the most preferred policy. Utility of type (D, θ) from Candidate (D, x) is $B - |\theta - x|$ and $-|\theta - x|$ from Candidate (R, x). Similarly, type (R, θ) also has θ as ideal point, but gets a utility benefit of B from the Republican candidate. However, this "utility function" is not a standard utility function in the sense that it completely describes behavior. In particular, they assume

[8] Implicitly, separability of preferences is assumed in several internet-based political comparison programs. For example, smartvote.ch (a cooperation project of several Swiss universities) collects the political positions of candidates in national elections by asking candidates a number of yes/no questions on different political issues. Voters can answer the same questions on a website (and also choose a weight for each issue) and are given a list of those candidates who agree with them most. Similar programs exist for the U.S. (http://www.myspace.com/mydebates), Germany (http://www.wahl-o-mat.de), Austria (http://www.wahlkabine.at/) and the Netherlands (http://www.stemwijzer.nl/).

that citizens abstain (i) if the utility difference between candidates is below a threshold ("abstention from indifference"), or (ii) if the utility from the preferred candidate is below some threshold T ("abstention from alienation"). While the model of Erikson and Romero (1990) has only the first effect and generates equilibrium convergence, the second effect may lead to (effective) preferences violating UCR. To see this, consider only the second effect, and define effective voter preferences of a Democratic partisan (D, θ) given policy platforms x_D and x_R as

$$D \succ R \Leftrightarrow B - |x_D - \theta| > -|x_R - \theta| \quad \text{and} \quad B - |x_D - \theta| > T,$$
$$R \succ D \Leftrightarrow B - |x_D - \theta| < -|x_R - \theta| \quad \text{and} \quad -|x_R - \theta| > T,$$
$$D \sim R \Leftrightarrow B - |x_D - \theta| \leqslant T \quad \text{and} \quad -|x_R - \theta| \leqslant T.$$

In order to have some participation, $B \geqslant T$ and in order for the alienation constraint to matter $B \leqslant T + 0.5$. To see that these preferences violate UCR, consider a Democratic partisan with an ideal policy point of $\theta = 0$. If both candidates were to propose the same policy $x_D = x_R = 0$, then $D \succ R$ (i.e., the voter votes for D). If, instead, $x = 0.5$ then $D \sim R$, because the voter is alienated and therefore abstains. Thus, these preferences violate UCR.[9]

Theorem 3 indicates that we have to focus on non-UCR preferences in order to generate policy divergence. In fact, it is easy to find such voting games.

Example 3. There are two candidates $c_G \neq c_B$ and two policies, a_G, a_B, where a_G is interpreted as focusing spending on national security (guns), while a_B corresponds to focusing on healthcare or schooling (butter). Candidate 0 is knowledgeable about national security issues, while Candidate 1's expertise is on social policies. Thus, it is reasonable to assume that there are the following types of voters:

Type G: $(c_G, a_G) \succ (c_B, a_B) \succ (c_G, a_B) \succ (c_B, a_G)$.
Type B: $(c_B, a_B) \succ (c_G, a_G) \succ (c_B, a_G) \succ (c_G, a_B)$.

Thus, type G voters prefer "guns" to "butter", and also have a preference for competent policy implementation, i.e., they prefer policies implemented by the candidate who has the corresponding expertise. Type B voters prefer "butter" to "guns", and also seek competence in policy implementation. Let the number of citizens of each type be given by $n_G(\omega)$ and $n_B(\omega)$, respectively, where $\omega \in \Omega$ reflects uncertainty about the distribution of preferences. Then the number of voters in state ω is given by

	(c_B, a_G)	(c_B, a_B)
(c_G, a_G)	$n_G(\omega) + n_B(\omega)$, 0	$n_G(\omega)$, $n_B(\omega)$
(c_G, a_B)	$n_G(\omega)$, $n_B(\omega)$	0, $n_G(\omega) + n_B(\omega)$

Then (a_G, a_B) is the unique Nash equilibrium and (c_G, a_G), (c_B, a_B) are the unique equilibrium platforms of the game with vote-share maximization. If, in addition, $\mu(\{\omega | n_G(\omega) > n_B(\omega)\}) > 0$ and $\mu(\{\omega | n_G(\omega) < n_B(\omega)\}) > 0$, then (c_G, a_G), (c_B, a_B) are also the equilibrium platforms of the game where candidates maximize the winning probability.[10]

Are UCR preferences a *necessary* condition for equilibrium policy convergence? It is easy to see that no property imposed solely on citizens' preferences, such as UCR, can be simultaneously necessary and sufficient for policy convergence. For example, if citizens with non-UCR preferences are never pivotal, then the violation of UCR would not matter for equilibrium convergence. The same is true if UCR is violated for some policies that are sufficiently undesirable for most voters. However, Theorem 4 shows that even if there is just one voter with arbitrary non-UCR preferences, then there are always some voting games in which everyone else has UCR preferences, but that have a strict equilibrium with policy divergence.

This is analogous to the well-known condition of single-peaked preferences in a one-dimensional policy space. If all voters have single-peaked preferences, the existence of a Condorcet winner is guaranteed. If some voters don't have single-peaked preferences, a Condorcet winner may still exist, but it is also possible to construct examples in which only one voter violates single-peakedness and no Condorcet winner exists.

Theorem 4. *Let $\succcurlyeq$ be some arbitrary non-UCR preferences on $C \times A$ and suppose that A is finite. Then there exists a voting game with the following property*:

[9] Since voters in Erikson and Romero (1990) and Adams and Merrill (2003) only fulfill transitivity for strict preferences, our theorems do not apply directly. However, from comparing the two models, it is clear that the violation of UCR in Adams and Merrill (2003) drives the divergence result.

[10] Note that we can easily add more voter types to Example 3 without immediately affecting the equilibrium. Even adding an arbitrary number of partisans (who vote for one candidate irrespective of the candidate's policy) preserves (c_G, a_G), (c_B, a_B) as the unique Nash equilibrium, as long as type G and B voters remain pivotal with positive probability. If the probability that type G and B voters are pivotal is zero, then any combination of strategies is an equilibrium of the game where candidates maximize their probability of winning.

Table 1
Construction of divergence equilibria with one non-UCR voter.

(a) Non-UCR voter			(b) UCR voters			(c) All voters		
	(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')
(c_0, a)	1, 0	0, 1	(c_0, a)	1, 1	1.5, 0.5	(c_0, a)	2, 1	1.5, 1.5
(c_0, a')	1, 0	0, 1	(c_0, a')	1.5, 1.5	1, 1	(c_0, a')	1.5, 1.5	1, 2

1. *One citizen has preferences $\succcurlyeq$ and all other citizens have UCR preferences.*
2. *There exists a pure strategy Nash equilibrium with policy divergence, for both winning probability and vote share maximization. Furthermore, the equilibrium is strict, and there is no other Nash equilibrium in either pure or mixed strategies.*

The detailed construction of the voting games is in the proof of Theorem 4 in Appendix A, but we provide an intuition based on a (generalizable) example here in which candidates maximize their vote share. Consider an individual whose preferences violate UCR for actions a and a'. There are just a few possibilities how candidate choices of policies a or a' translate into votes for the candidates. Since the preferences violate UCR, the diagonal elements cannot be same. For example, our non-UCR voter's voting behavior for actions a and a' of the candidates could be the one summarized in Table 1. The numbers in this table denote the votes for the two candidates, for example, "1, 0" denotes that the non-UCR voter votes for Candidate 0.

Of course, A may consist of more than just the two policies a, a' and the non-UCR voter may strictly prefer some other policies, in which case the violation of UCR for a and a' may be irrelevant. In order to make a and a' relevant, we introduce two additional voters who prefer a and a' to all other policies. Suppose one voter prefers Candidate 0 while the other prefers Candidate 1 if both candidates choose the same policy. UCR does not impose any restriction on the choice of off-diagonal elements. Allowing for some uncertainty about the state of the world, we can generate the vote shares given in Table 1b from the two UCR-voters. Note that, if candidates were only to play for the support of the two UCR-voters, then the unique Nash equilibrium (a, a) is on the diagonal, and therefore involves policy convergence.

Now add all three voters together (Table 1c). Note that, without loss, we can exclude actions other than a and a' since the two UCR voters rank those below a and a'.[11] Now a, a' is a strict and unique Nash equilibrium. We provide a similar construction of the game with winning-probability maximization.

The example is robust in the sense that we could change the preferences of the two UCR-voters somewhat and still obtain the same result. What matters is primarily that the policies for which UCR is violated for agent 1 are among the most preferred policies of sufficiently many of the UCR-voters so that candidates want to use them. Second, if the candidates' objective is to maximize their winning probability, then the non-UCR voter must be pivotal for the election outcome with positive probability. These are the crucial aspects of the construction of the example, while everything else can easily be changed without affecting the conclusion that the unique equilibrium features divergence.

5. A generalized probabilistic voting model

Another way of showing that UCR is "close" to a necessary condition for policy convergence is to restrict attention to a parametrized class of preferences, and prove that UCR is necessary and sufficient for convergence in voting games when voters have preferences within this class. We adopt this approach in this section.

5.1. The classic model with microfoundation

In the classical probabilistic voting model (PVM), groups are identified as voters with the same "economic" preferences. However, voters within the same group may vote for different candidates because of what Persson and Tabellini (2000), p. 52 refer to as "ideology". They write that "one way to motivate [ideology] is to think about a second policy dimension, orthogonal to fiscal policy, in which candidates cannot make credible commitments, but set an optimal policy after the election according to their ideology". Rather than modeling the second policy dimension explicitly, they operationalize this idea by adding an additive ideology shock to the economic preferences.

Our objective here is to setup a model that takes this notion of a fixed second policy dimension seriously. We start with the special case of Euclidean preferences in a two-dimensional policy space, i.e., circular indifference curves. In the following section, we analyze a model in which indifference curves can take any elliptical form, and which captures the notion of complementarity between the two dimensions.

Suppose that voters have one of finitely many policy ideal points θ_j, $j = 1, \ldots, J$. Let λ_j be the fraction of voters with ideal point θ_j. We assume that λ is deterministic. Voters with policy preference θ_j are differentiated with respect to their ideal point on the fixed issue. The distribution of ideal points on fixed issues, δ, for voters in group j is given by the cdf $F_j(\cdot)$ with corresponding pdf $f_j(\cdot)$. In Sections 5.1 and 5.2 we focus on the case of vote-share maximization, but we show in Section 5.5 that the model can easily be adjusted for candidates who maximize the probability of winning.

[11] Thus, a candidate who were to propose another policy would always lose against one who proposes either a or a'.

Let $c_i \in \mathbb{R}$ be Candidate i's fixed position, and $a \in \mathbb{R}$ the flexible policy dimension. Suppose preferences are Euclidean with utility function $u_{\delta,\theta}(a,c) = -(\delta - c)^2 - (\theta - a)^2$. These preferences are of the type represented in (2) and hence UCR.

A type j voter with ideal point δ_j on the fixed issue prefers Candidate 0 to Candidate 1 if and only if

$$(\delta_j - c_0)^2 + (\theta_j - a_0)^2 < (\delta_j - c_1)^2 + (\theta_j - a_1)^2. \tag{3}$$

Eq. (3) is equivalent to

$$\delta_j < \frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right]. \tag{4}$$

The fraction of voters who support Candidate 0 is therefore given by

$$\sum_{j=1}^{J} \lambda_j F_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right]\right). \tag{5}$$

We use the following assumption in the current and the following subsection.

Assumption 1. f_j is continuously differentiable, and $f_j'(0.5(c_0 + c_1)) = 0$.

Continuous differentiability is a fairly innocuous technical assumption, and $f_j'(0.5(c_0 + c_1)) = 0$ is a sufficient condition that ensures that the second-order condition is satisfied and bounded away from zero (so it is clear that this condition could be relaxed at the expense of more cumbersome algebra).

In order to maximize his vote share, Candidate 0 maximizes (5) over a_0, while Candidate 1 minimizes (5) over a_1. The first-order conditions of the candidates' respective optimization problems are

$$\sum_{j=1}^{J} \lambda_j f_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right]\right)\frac{\theta_j - a_0}{c_1 - c_0} = 0; \tag{6}$$

$$-\sum_{j=1}^{J} \lambda_j f_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right]\right)\frac{\theta_j - a_1}{c_1 - c_0} = 0. \tag{7}$$

Adding (6) and (7) gives

$$\frac{a_1 - a_0}{c_1 - c_0}\sum_{j=1}^{J} \lambda_j f_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right]\right) = 0,$$

which implies that any solution satisfies $a_0 = a_1$. Substituting a_0 for a_1 in (6) and solving yields

$$a_0 = a_1 = \frac{\sum_{j=1}^{J} \lambda_j f_j(\frac{c_0+c_1}{2})\theta_j}{\sum_{j=1}^{J} \lambda_j f_j(\frac{c_0+c_1}{2})}. \tag{8}$$

Formula (8) the equilibrium condition in the PVM. In particular, the optimal policy is a weighted average over the candidates' position, where the weights are determined by the distribution of types.

The second-order conditions of the candidates' optimization problems, evaluated at $a_0 = a_1$ are identical, and given by

$$\sum_{j=1}^{J} \lambda_j\left(\frac{(\theta_j - a_0)^2}{c_1 - c_0}\frac{f'(\frac{c_0+c_1}{2})}{f(\frac{c_0+c_1}{2})} - 1\right) < 0. \tag{9}$$

Since $f'(0.5(c_0 + c_1)) = 0$, condition (9) is satisfied. Thus, $a_0 = a_1$ is a local, strict equilibrium, where a "local equilibrium" is the following standard concept:

Definition 3. A pair of strategies (a_0, a_1) is a *local equilibrium* if there exist sets $\tilde{A}_0$ and $\tilde{A}_1$ such that $a_0 \in \operatorname{int}(\tilde{A}_0)$ and $a_1 \in \operatorname{int}(\tilde{A}_1)$, and (a_0, a_1) is a Nash equilibrium of the game in which candidates are restricted to choose from $\tilde{A}_0$ and $\tilde{A}_1$, respectively.

Sufficient conditions for global optimality are difficult to state, as the left-hand side of (9) can be positive if f' is evaluated sufficiently far away from $\frac{c_0+c_1}{2}$; see Banks and Duggan (2005) for a general treatment of existence problems in the classical probabilistic voting model. However, if we restrict a to be from a sufficiently small interval $[\underline{\alpha}, \bar{\alpha}]$ that contains $a_0 = a_1$, then the local equilibrium that we identified is also guaranteed to be a global equilibrium in the restricted game. Theorem 3 therefore implies (corresponding to standard results for the standard PVM with additive ideology shocks) that a_0, a_1 is the unique Nash equilibrium, pure or mixed, of the restricted game.

Theorem 5. *Suppose that Assumption 1 is satisfied. Then (a_0, a_1) given by (8) is a strict local Nash equilibrium with policy convergence, i.e., $a_0 = a_1$. Furthermore, the equilibrium is the unique local pure strategy Nash equilibrium in the original (unrestricted) game.*

As in the standard PVM, the interval $[\underline{\alpha}, \bar{\alpha}]$ becomes larger (or global), if the type distribution is more spread out, i.e., if f' stays small if we move away from zero. Of course, if $f' \equiv 0$ (i.e., if the distribution is uniform) then the equilibrium is always global.

5.2. Elliptical preferences

We now consider preferences for which indifference curves are ellipses rather than circles. Intuitively, indifference curves that are circles capture preferences where the ideal policy a is independent of the fixed characteristic c. In contrast, consider, for example, elliptical indifference curves for which the major axis is the 45 degree line. This corresponds to a situation where the fixed characteristic and the policy are complements in the sense that, the higher is the candidate's fixed characteristic, the higher is the voter's ideal policy a.

For example, consider the following situation: The fixed characteristic measures the general attitude of the candidate towards cooperation with foreign governments in solving international security problems. A candidate who favors broad international cooperation and consensus building in international organizations would be denoted as (say) a low type on this dimension, while a candidate who prefers a unilateral approach and does not care much about the international opinion would be a high type. Candidates are fixed to their respective (different) positions in that dimension. This assumption appears to be reasonable, as it is probably very difficult to credibly commit to a particular foreign policy "attitude".

There is a second dimension that is more concrete and where candidates can commit to a particular position. For concreteness, think of this dimension as the defense budget. It is quite plausible that the type of the executive (i.e., the position of a candidate in the first dimension) influences a voter's preferences over policy in the second dimension; for example, a voter may prefer that a more assertive candidate has a higher (or lower) defense budget than a more cooperative type. In the first case, we would say that characteristic and policy are complements, in the second case, they are substitutes. Both cases imply that a voter's indifference curves are not circles but rather could be captured by ellipses whose major axis is not exactly horizontal or vertical.[12]

Before we proceed, it is useful to conceptually differentiate between the shape of the indifference curves and correlation in the distribution of ideal points. So far, we have argued that it is plausible that a single voter's preferences over fixed characteristics and flexible policies display complementarity or substitutability. This effect influences the shape of indifference curves. Conceptually different from this is correlation in the distribution of ideal points in both dimensions. For example, it may be the case that many voters who have a preference for "tough-talking" executives also have, on average, a higher ideal point on the defense budget. Thus, if we were to plot voter ideal points in a $(c - a)$-diagram, these ideal points might display positive correlation. Whether or not there is correlation in ideal points does not affect our theory much, so we do not need to take a position on this question.

Consider the preferences illustrated in Fig. 1 where the two parameters κ_1 and κ_2 determine the shape of the indifference curves (κ_1 measures the ratio of the two axes, while κ_2 is the angle of rotation). Clearly, any preferences with elliptical indifference curves can be represented by the ideal point (δ, θ), κ_1 and κ_2. In particular, letting $\kappa_1 = 1$ produces standard Euclidean preferences, reducing the model to the standard PVM.

More formally, let

$$M = \begin{pmatrix} 1 & \kappa_2 \\ -\kappa_1\kappa_2 & \kappa_1 \end{pmatrix}. \tag{10}$$

For $x \in [0,1]^2$ define the norm $\|x\|_M = \|Mx\|_2$, where $\|\cdot\|_2$ denotes the Euclidean norm. Let (δ, θ) be a voter's ideal point. Then

$$(c, a) \succcurlyeq^{\delta,\theta} (c', a') \quad \text{if and only if} \quad \|(c, a) - (\delta, \theta)\|_M \leqslant \|(c', a') - (\delta, \theta)\|_M. \tag{11}$$

It is easy to check that indifference curves are of the form

$$(c - \delta, a - \theta) \begin{pmatrix} 1 + \kappa_1^2\kappa_2^2 & \kappa_2(1 - \kappa_1^2) \\ \kappa_2(1 - \kappa_1^2) & \kappa_1^2 + \kappa_2^2 \end{pmatrix} \begin{pmatrix} c - \delta \\ a - \theta \end{pmatrix} = \bar{u}. \tag{12}$$

[12] As a related example where complementarity between a candidate's type and the policy choice is plausible, consider the following example: Suppose candidates differ with respect to their beliefs about the possibility of rehabilitating criminal convicts. While a low c candidate believes that rehabilitation is often effective, a high c candidate believes that it does not. Consequently, if the tough politician is in power, criminals will remain more or less unreformed (whether or not rehabilitation is in principle possible). Suppose that a corresponds to the amount of money spent on building and maintaining prisons (not including any rehabilitation expenses). Then, independent of their ideal point, voters would want the candidate who does not believe in rehabilitation to build more prisons, since absent rehabilitation efforts, this is the better choice than releasing prisoners early because of a lack of space in prisons. In contrast, if the executive believes in and funds rehabilitation programs, additional prison space is less useful, and the voter would prefer a lower a.

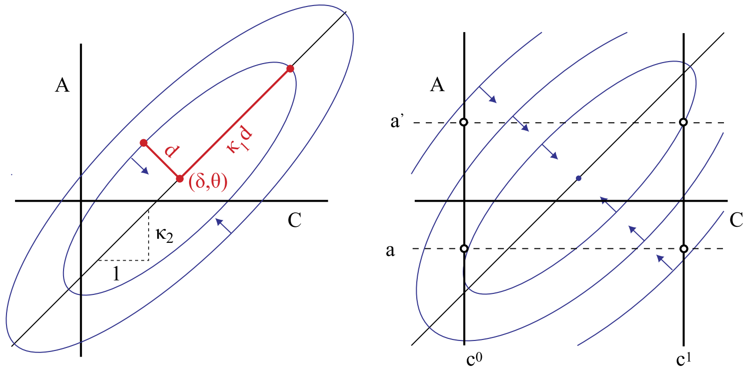

Fig. 1. Elliptical preferences and violation of UCR.

The eigenvectors of the matrix in (12) are $(-\kappa_2, 1)$ and $(1, \kappa_2)$ with associated eigenvalues $\kappa_1^2(1+\kappa_2^2)$ and $1+\kappa_2^2$. Thus, as indicated in Fig. 1 indifference curves are elliptical, with the main axes given by the above eigenvectors, and the ratio of the length of the axes measured by κ_1.

If the major axis has positive slope such as in the left panel, then a voter's optimal level of a increases with c, and we say that c and a are complements. If, in contrast, the slope of the major axis is negative, we say that c and a are substitutes. Formally, if $u(c,a) = -\|(c,a) - (\delta,\theta)\|_M^2$ represents the preferences, then

$$\frac{\partial^2 u(c,a)}{\partial c \partial a} = -2\bigl(1-\kappa_1^2\bigr)\kappa_2. \tag{13}$$

For $\kappa_1 > 1$ and $\kappa_2 > 0$ as in the graph, the sign of the cross derivative is positive, indicating complements.

We next prove that UCR is violated for these preferences. The violation of UCR can most easily be seen in the right panel of Fig. 1 (and the argument can clearly be formalized). If both candidates select policy a then the voter prefers Candidate 0. If, instead, both candidates select policy a' then the voter prefers Candidate 1. The only elliptical preferences that satisfy UCR are those for which the major or minor axis is horizontal, i.e., where $\kappa_2 = 0$. Such preferences are given by a utility function $u(c,a) = -k^2(c-\delta)^2 - (a-\theta)^2$. In this case, $u(c,a) \geqslant u(c',a)$ if and only if $u(c,a') \geqslant u(c',a')$.

Directly analyzing the voting game with elliptical indifference curves would be very complicated. Thus, we transform the policy space such that preferences become Euclidean (in the transformed model) and thus satisfy UCR. Theorem 3 can then be used to identify possible equilibria and to prove uniqueness of equilibrium.

We now use Fig. 2 to explain this procedure. The detailed mathematical arguments can be found in Appendix A. The top left panel of Fig. 2 depicts the original model. In the standard PVM, individuals with the same θ are interpreted as a "group" that has the same "economic" interests (i.e., ideal value of policy). Members of the same group differ only in their "ideological" preferences captured by δ (i.e., their ideal value of the fixed position). In PVMs, it is standard to consider finitely many "groups" (each with a continuous, possibly group-specific distribution of ideology), and we adopt the same approach. In Fig. 2, there are three "groups" with policy ideal points θ_1, θ_2 and θ_3, and the indifference curves of one particular type with a policy ideal point of θ_3. We apply a linear transformation (given by matrix M in (10) above) to the top left panel. As indicated, the x and y-axes coincide with the directions of the major and minor axes of the ellipses. We apply a rotation, indicated by the curved clockwise arrow, and at the same time we stretch along the y-axis as indicated by the straight arrow pointing northwest until indifference curves become circles. The result of applying M is depicted in the top right-panel. Note that the x and y-axes are now horizontal and vertical, while the locus of voter types as well as the set of feasible policy are skew and no longer form a right angle (because of the stretching).

It is more convenient to analyze the model in the two positions depicted in the middle panels. Both are obtained by applying rotations to the top right panel. In the middle left panel, the candidates' sets of feasible policies are vertical lines (and the indifference curves are circles and therefore satisfy UCR). As a consequence, Theorem 3 applies that in any strict Nash equilibrium policies must be identical, i.e., $a_0 = a_1$. If an equilibrium exists, then second-order conditions guarantee strictness, just like in the standard PVM. Thus, if an equilibrium exists, it must also be unique.

Existence can be shown most easily using the right-middle panel. This corresponds to the PVM from the previous section, except that the candidates' feasible policy lines are skew. As indicated in the graph, the slope of the policy lines is given by $1/\beta$, where

$$\beta = \frac{\kappa_2(1-\kappa_1^2)}{\sqrt{1+\kappa_1^2\kappa_2^2}}. \tag{14}$$

Note that β has exactly the opposite sign of (13). Thus, if c and a are complements as in Fig. 2, then $\beta < 0$.

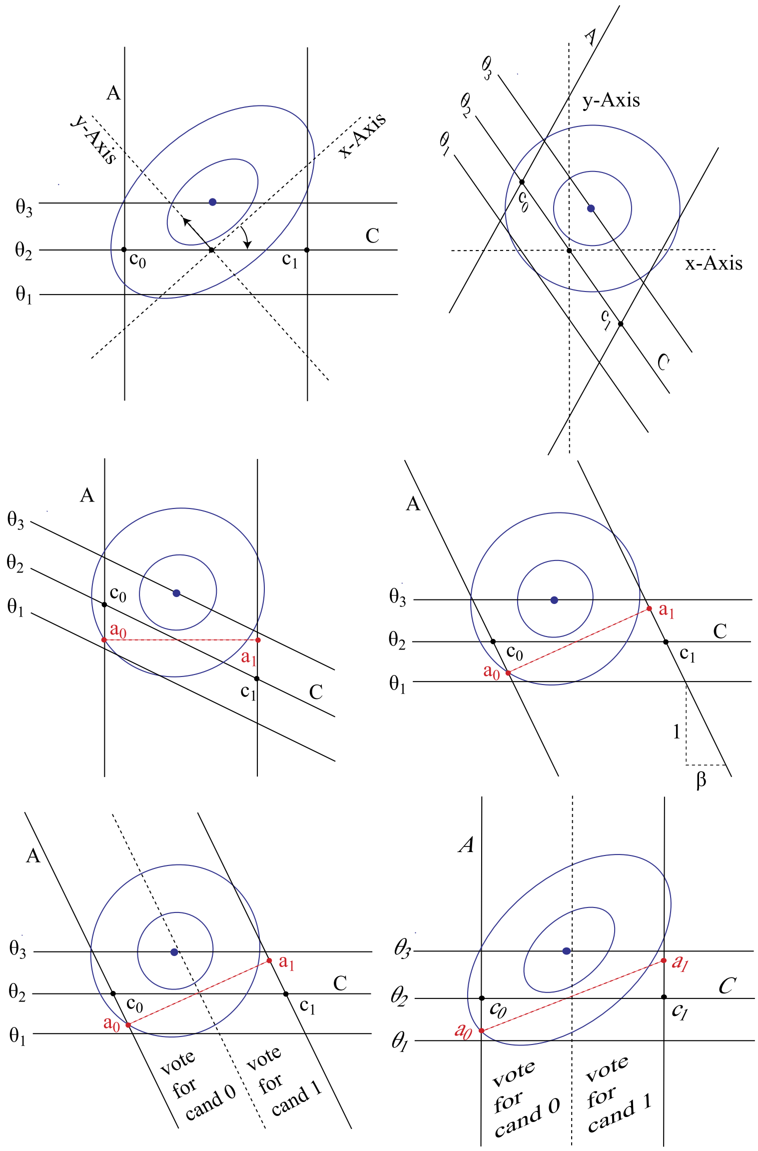

Fig. 2. Transforming elliptical preferences to Euclidean preferences and equilibrium.

If the main axes of the ellipses in the original mode are horizontal or vertical, i.e., if $\kappa_2 = 0$, or if indifference curves are circles at the outset, i.e., $\kappa_1 = 1$, then $\beta = 0$. In this case, the two middle panels are identical, and as a consequence, $a_0 = a_1$, i.e., there is policy convergence.

Now return to the case of $\beta \neq 0$. As we rotate the graph from the middle left panel to the middle right panel, the condition $a_0 = a_1$ becomes

$$\tilde{a}_1 = \tilde{a}_0 - \frac{\beta}{\beta^2 + 1}(\tilde{c}_1 - \tilde{c}_0), \tag{15}$$

where the tilde above each parameter indicates that coordinates are with respect to the axis in the middle right panel. Eq. (15) and Fig. 2 imply $\tilde{a}_0 \neq \tilde{a}_1$ (note that Theorem 3 does not apply in the middle right panel, since the candidates' feasible policy lines are skew).

To determine the necessary and sufficient conditions for equilibrium we proceed as in the previous section, except that we need to adjust for the fact that the feasible policy lines are skew. The resulting first-order conditions are

$$\sum_{j=1}^{J} \lambda_j f_j\left(\frac{\tilde{c}_0 + \tilde{c}_1}{2\sqrt{1 + \kappa_1^2 \kappa_2^2}}\right)\left[\frac{\beta}{2} + \left(1 + \beta^2\right)\frac{\tilde{\theta}_j - \tilde{a}_0}{\tilde{c}_1 - \tilde{c}_0}\right] = 0; \tag{16}$$

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{\tilde{c}_0+\tilde{c}_1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\right)\left[-\frac{\beta}{2}+\left(1+\beta^2\right)\frac{\tilde{\theta}_j-\tilde{a}_1}{\tilde{c}_1-\tilde{c}_0}\right]=0. \tag{17}$$

The second-order conditions, detailed in (51) and (52) are of the form

$$\sum_{j=1}^{J}\lambda_j\left(\frac{\Gamma_i(\tilde{c}_0,\tilde{c}_1,\tilde{a}_0,\tilde{a}_1,\theta_j)}{2(1+\beta^2)(\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0))}\frac{f'(\cdot)}{f(\cdot)}-1\right)<0,$$

where Γ_i is a function of the indicated variables, and the Candidate $i=0,1$. One can check that, at the solution of the first-order conditions, $f'(\cdot)=0$. Hence, the second-order conditions are satisfied, and we have again at least a local equilibrium.

Finally, we transform the policy space back into its original form. This process is illustrated in the bottom panel of Fig. 2. After the transformation, policies still differ. The line separating supporters for Candidates 0 and 1 is vertical as in the standard model. The first-order conditions (16) and (17) change to

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{c_0+c_1}{2}\right)\left[\frac{\beta}{2}+\frac{(1+\beta^2)\kappa_1(1+\kappa_2^2)}{1+\kappa_1^2\kappa_2^2}\frac{\theta_j-a_0}{c_1-c_0}\right]=0; \tag{18}$$

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{c_0+c_1}{2}\right)\left[-\frac{\beta}{2}+\frac{(1+\beta^2)\kappa_1(1+\kappa_2^2)}{1+\kappa_1^2\kappa_2^2}\frac{\theta_j-a_1}{c_1-c_0}\right]=0. \tag{19}$$

Condition (15) changes to

$$a_1-a_0=-\frac{\beta(1+\kappa_1^2\kappa_2^2)}{(1+\beta^2)\kappa_1(1+\kappa_2^2)}(c_1-c_0). \tag{20}$$

Thus, if $\beta<0$, which is the case of complementarity between fixed characteristics and policy depicted in Fig. 2, then $a_0<a_1$ as the right-hand side of (20) is positive in the case. If, instead, $\beta>0$ then fixed characteristics and policy are substitutes, and $a_0>a_1$.

The arithmetic average $\bar{a}=(a_0+a_1)/2$ of policies a_0 and a_1 has no direct substantive significance in our model (in particular, it is not the *expected* policy, as the candidates' winning probabilities are usually different). However, we can use $\bar{a}$ to show uniqueness of local equilibria as follows. Add Eqs. (18) and (19) to get

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{c_0+c_1}{2}\right)\left[\frac{2(1+\beta^2)\kappa_1(1+\kappa_2^2)}{1+\kappa_1^2\kappa_2^2}\frac{\theta_j-\bar{a}}{c_1-c_0}\right]=0. \tag{21}$$

Since the coefficient of $\frac{\theta_j-\bar{a}}{c_1-c_0}$ is strictly positive, (21) simplifies to

$$\sum_{j=1}^{J}\lambda_j f_j\left(\left(\frac{c_0+c_1}{2}\right)\right)(\theta_j-\bar{a})=0, \tag{22}$$

which is identical to (8) (replacing a_0 by $\bar{a}$). Thus, $\bar{a}$ is exactly the same as the equilibrium policy in a Euclidean model where $\kappa_1=1$ or $\kappa_2=0$.[13]

We now summarize our results. It should be noted that the requirements for existence in Theorem 6 mirror those in Theorem 5 and thus correspond to those in the standard PVM.

Theorem 6. *Suppose that Assumption 1 is satisfied and that preferences are given by (11). Then there exists a strict local Nash equilibrium (a_0,a_1) where a_0 and a_1 satisfy (20). There is policy divergence, i.e., $a_0\neq a_1$, unless indifference curves are circles or the major axis is horizontal or vertical. Moreover, there does not exist any other local pure strategy Nash equilibrium.*

5.3. Comparison of the classic and the general spatial models

One of the main points of interest of the standard PVM is to determine which features of the distribution of voter preferences influence the equilibrium policy. The central finding of the PVM is that the equilibrium policy maximizes a weighted sum of the voters' economic (i.e., non ideological) utilities, $-(\theta_j-a)^2$, where the weights of group j in the maximization problem is determined both by the group size λ_j, and by how many members of group j can be moved

[13] In fact, $\bar{a}$ is uniquely determined, and this fact can be used to provide an alternative proof for uniqueness of a local equilibrium.

easily, which is determined by $f_j(\cdot)$. The same determinants influence equilibrium policy in the general spatial model. In particular, policy $\bar{a}$ solves exactly the same optimization problem, and existence of equilibrium can be proved along the same lines as in the standard model (once the setting is transformed as explained in the previous section).

The key difference between the classical and the general spatial models is that, in the classical model, both candidates solve the same optimization problem and thus their equilibrium policies coincide. In contrast, the optimization problems of the two candidates differ with general preferences, resulting in policy divergence. The extent of policy divergence increases in the ex-ante difference between candidates. In practice, the ex-ante differences between candidates may increase if parties are more polarized on the dimension captured by the fixed characteristic, c. In contrast, the difference between candidates' fixed characteristics are irrelevant for policy choice in the standard model.

5.4. General elliptical preferences

The results of Section 5.2 can be generalized to the case where both C and A are multidimensional. In particular, suppose that $C = \mathbb{R}^k$ and $A = \mathbb{R}^m$, and let $n = k + m$. As above, we consider a finite collection of groups $\theta_j \in \mathbb{R}^m$, $j = 1, \dots, K$. For each group j, there is a cumulative distribution $F_j(\delta)$ on C.

If preferences are Euclidean, then a voter with ideal position $(\delta, \theta_j) \in \mathbb{R}^n$ prefers Candidate 0 to Candidate 1 if $\|(\delta, \theta_j) - (c_0, a_0)\|_2 < \|(\delta_j, \theta_j) - (c_1, a_1)\|_2$, where $(c_i, a_i) \in \mathbb{R}^n$ is Candidate i's position (including the fixed characteristics). This condition, which generalizes (4), can be rewritten as

$$\sum_{i=1}^{k} \delta_{j,i}(c_{1,i} - c_{0,i}) \leqslant \frac{1}{2}\left[\sum_{i=1}^{m}\big((\theta_{j,i} - a_{1,i})^2 - (\theta_{j,i} - a_{0,i})^2\big) + \sum_{i=1}^{k}\big(c_{1,i}^2 - c_{0,i}^2\big)\right]. \tag{23}$$

Let $\hat{F}_j$ be the distribution of $\sum_{i=1}^{k} \delta_{j,i}(c_{1,i} - c_{0,i})$. Then Candidate 0's vote share is

$$\sum_{j=1}^{J} \lambda_j \hat{F}_j\left(\frac{1}{2}\left[\sum_{i=1}^{m}\big((\theta_{j,i} - a_{1,i})^2 - (\theta_{j,i} - a_{0,i})^2\big) + \sum_{i=1}^{k}\big(c_{1,i}^2 - c_{0,i}^2\big)\right]\right), \tag{24}$$

which is the analogue of (5). Candidate 0 chooses $a_0 \in A$ to maximize (24), while Candidate 1 chooses $a_1 \in A$ to minimize it. By arguments analogous to those in Section 5.1, $a_0 = a_1$ in the local equilibrium.

In order to generalize this model to elliptical preferences, let O be an arbitrary orthogonal $(n \times n)$-matrix, and let D be an $(n \times n)$-diagonal matrix with diagonal entries $d_i > 0$, $i = 1, \dots, n$. Let $M = D \cdot O$. Then, as in Section 5.2, define the norm $\|x\|_M = \|Mx\|_2$. A voter with ideal point (δ, θ) now prefers Candidate 0 to Candidate 1 if $\|(\delta, \theta_j) - (c_0, a_0)\|_M < \|(\delta_j, \theta_j) - (c_1, a_1)\|_M$.

Let $\langle x, y\rangle$ be the inner product of two vectors x and y. For arbitrary $z \in \mathbb{R}^n$ we get

$$\begin{aligned}\|z\|_M^2 &= \|Mz\|_2^2 = \langle Mz, Mz\rangle = \langle D \cdot Oz, D \cdot Oz\rangle = \big\langle z, (D \cdot O)^t \cdot D \cdot Oz\big\rangle \\ &= \big\langle z, O^t \cdot D^t \cdot D \cdot Oz\big\rangle = \big\langle z, O^{-1} \cdot D^2 \cdot Oz\big\rangle = z^t O^{-1} \cdot D^2 \cdot Oz.\end{aligned} \tag{25}$$

Note that $O^t = O^{-1}$ because O is orthogonal, and $D^t = D$ since D is a diagonal matrix. Thus, $A = O^{-1} \cdot D^2 \cdot O$ is a self-adjoint matrix. The eigenvalues are given by d_i^2, the squares of the diagonal elements of D, and the eigenvectors are given by $O^{-1}e_i$, where e_i is the ith unit vector. To see this, note that $A \cdot O^{-1}e_i = O^{-1} \cdot D^2 e_i = O^{-1} d_i^2 e_i = d_i^2 O^{-1} e_i$.

Thus, the quadratic form (25) describes a utility function with elliptical indifference contours in a multidimensional space. The directions of the main axes are given by the above eigenvectors, and the length of the axes are proportional to the eigenvalues, as in Section 5.2. Again, we can apply matrix $M = D \cdot O$ to transform the elliptical preferences into Euclidean preferences.[14] The set of feasible policies of candidates 0 and 1 are given by the parallel lines

$$L_0 = \left\{M\begin{pmatrix} c_0 \\ a \end{pmatrix} \,\middle|\, a \in \mathbb{R}^m\right\} \quad \text{and} \quad L_1 = \left\{M\begin{pmatrix} c_1 \\ a \end{pmatrix} \,\middle|\, a \in \mathbb{R}^m\right\}.$$

Since preferences are Euclidean and therefore UCR, the local equilibrium consists of points $z_i \in L_i$ such that $z_1 - z_0$ is orthogonal to L_0 and L_1 (see Fig. 2).

In order for a vector x to be orthogonal to L_1 and L_2, $0 = \langle M\binom{0}{a}, x\rangle = \langle\binom{0}{a}, O^{-1} \cdot Dx\rangle$, for all $a \in \mathbb{R}^m$, which implies $x = D^{-1} \cdot O\binom{c}{0}$, for some $c \in \mathbb{R}^k$. Thus, the equilibrium policies a_0 and a_1 must satisfy $M\binom{c_0}{a_0} + D^{-1} \cdot O\binom{c}{0} = M\binom{c_1}{a_1}$, which implies

[14] To generate formula (10) from Section 5.2 using D and O, simply define

$$D = \sqrt{1 + \kappa_2^2}\begin{pmatrix} 1 & 0 \\ 0 & \kappa_1 \end{pmatrix} \quad \text{and} \quad O = \frac{1}{\sqrt{1 + \kappa_2^2}}\begin{pmatrix} 1 & \kappa_2 \\ -\kappa_2 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} c_1 \\ a_1 \end{pmatrix} = \begin{pmatrix} c_0 \\ a_0 \end{pmatrix} + O^{-1} \cdot D^{-2} \cdot O \begin{pmatrix} c \\ 0 \end{pmatrix}. \tag{26}$$

Generalizing the argument from Section 5.2, for preferences to be UCR, there are two possibilities: First, the indifference curves can be circles (balls), which happens when D is the identity matrix I, in which case $O^{-1} \cdot D^{-2} \cdot O = I$. Second, the main axes of the ellipses coincide with the coordinate axes, in which case $O^{-1} \cdot D^{-2} \cdot O$ is a diagonal matrix. In both cases, $c = c_1 - c_0$ and $a_0 = a_1$.

For general O and D, however, $O^{-1} \cdot D^{-2} \cdot O\binom{c}{0}$ will no longer be in the linear subspace $\{(c, 0) \mid c \in \mathbb{R}^k\}$ of $\mathbb{R}^m$, and as a consequence $a_0 \neq a_1$, i.e., we have policy divergence whenever preferences are not UCR.

Existence of local equilibria follows along the same lines as in the two-dimensional case, i.e., we must ensure that the density function of $\hat{F}$ is sufficiently spread out (i.e., its derivative is not too large).

5.5. The model with winning probability maximization

In Sections 5.1, 5.2 and 5.4 we have assumed that candidates maximize their vote share. We now sketch how the model can be modified if candidates maximize their probability of winning the election.

Suppose that the distribution of ideal points on fixed issues, δ, depends on a state ω. For given ω, the fraction of voters in group j of type δ or less is given by cdf $F_j(\delta - \omega)$, and the corresponding pdf is $f_j(\delta - \omega)$. Note that ω is a parameter that uniformly shifts the distribution of voters, e.g., a higher value of ω shifts the fixed-issue ideal points of all voters to the right. As in the general model, ω is distributed according to a probability distribution μ. Furthermore, remember that ω captures *all* of the uncertainty in our model: Given ω, we know what is the actual distribution of voter ideal points; for example, if $\omega = 0.5$, then $F_j(\delta - 0.5)$ measures the *realized proportion* of voters whose fixed issue ideal point is below δ (i.e., there is no two-stage uncertainty in the sense that individual voters' ideal points would be *drawn* from $f_j(\delta - 0.5)$ in this example).

We modify Assumption 1 as follows. In addition to $f_j'((c_0 + c_1)/2) = 0$ we also require that the median is located at $(c_0 + c_1)/2$, i.e., $F_j((c_0 + c_1)/2) = 0.5$ for all $j = 1, \ldots, J$. Further, ω has a distribution with strictly positive density on its support, and the support is a non-empty interval.

Consider the case with Euclidean preferences. (5) implies that for a given value of ω, the fraction of voters who support Candidate 0 is given by

$$\sum_{j=1}^{J} \lambda_j F_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right] - \omega\right). \tag{27}$$

Clearly, (27) is continuous and decreasing in ω, and goes to 0 for $\omega \to \infty$, while it goes to 1 for $\omega \to -\infty$. Thus, for any pair of policies (a_0, a_1), there exists a critical value $\omega^*(a_0, a_1)$ such that the election ends in a tie if $\omega = \omega^*(a_0, a_1)$. If $\omega < \omega^*$ then Candidate 0's wins because his vote share strictly exceeds 50%. The reverse is true, i.e., Candidate 1 wins, if $\omega \geqslant \omega^*$.

Since the election ends in a tie in state ω^* we have

$$\sum_{j=1}^{J} \lambda_j F_j\left(\frac{1}{2}\left[c_0 + c_1 + \frac{(a_1 - a_0)(a_1 + a_0 - 2\theta_j)}{c_1 - c_0}\right] - \omega^*\right) = 0.5. \tag{28}$$

The first-order conditions again imply that $a_0 = a_1$. Thus, (28) and the assumption that $F_j(0.5(c_0 + c_1)/2) = 0.5$ imply that the critical state is $\omega^* = 0$.

The second-order condition of the model with vote share maximization (9) becomes

$$\sum_{j=1}^{J} \lambda_j \left(\frac{(\theta_j - a_0)^2}{c_1 - c_0} \frac{f'\left(\frac{c_0+c_1}{2} - \omega^*\right)}{f\left(\frac{c_0+c_1}{2} - \omega^*\right)} - 1\right) < 0 \tag{29}$$

and, since $\omega^* = 0$, this is equivalent to (9).

The analysis of the model with elliptical preferences and winning probability maximization follows in a similar way from the vote share maximization model. Again the term $-\omega^*$ must be added as an argument to the voter type distributions F_j and f_j, respectively. However, it follows again that $\omega^* = 0$, and therefore all first- and second-order conditions are identical to those of the model where candidates maximize their vote share.

6. Conclusion

In this paper, we develop a model of candidate competition that is more general than the previous literature on this subject, as we allow for voters to care about both the candidates' fixed characteristics and their chosen policy platforms in an arbitrary way. The framework thus contains all existing frameworks of candidate competition — such as the spatial model or the probabilistic voting model — as special cases. Also, by interpreting some "core convictions" of the candidates as fixed characteristics, while candidates can freely choose their positions on other issues, our model provides a bridge between the classical Downsian model in which candidates can choose their platform without any restrictions, and the citizen candidate model in which candidates cannot commit to any policy that is not their ideal policy.

The main contribution of the model is twofold. First, it enhances our understanding of what drives certain features of equilibrium in existing models of candidate competition, notably policy convergence. Specifically, we show that assuming candidates are office-motivated and compete with each other does not, by itself, produce policy convergence. Rather, this conclusion follows from the interplay of office motivation and a certain "independence" of fixed characteristics and flexible policy positions in the voters' utility functions. We formalize this form of "independence" by identifying the class of UCR preferences for which equilibrium policy convergence arises even when candidates differ in fixed characteristics (Theorem 3). Conversely, Theorem 4 shows that UCR preferences are also, in a certain sense, necessary for convergence: Even if only one voter has non-UCR preferences, there exists a voting game in which the unique and strict Nash equilibrium features policy divergence.

For the most general setup, we obtain characterization results — they tell us how an equilibrium looks *if it exists*. Since our model contains a very general class of models, including some for which no pure strategy equilibrium exists, it is effectively impossible to identify necessary and sufficient conditions that guarantee existence of a pure strategy strict equilibrium within the general framework. Nevertheless, we know from previous literature that an equilibrium exists for several subclasses such as the one-dimensional spatial model and the probabilistic voting model. Thus, our characterization results are not vacuous, and they help us to understand *why* policy convergence obtains in these models.

The second major contribution of our paper is to identify an interesting class of models in which a candidate's competence in a policy area affects the voter's preferred policy from the candidate, which yields non-UCR preferences in a natural way. The model that we present captures the notions of complementarity between fixed and flexible positions, and is a generalization of the probabilistic voting model. The model is essentially as tractable as the probabilistic voting model in that there is (at least under certain additional, relatively mild conditions) a unique and strict Nash equilibrium that can easily be characterized. However, we show that the equilibrium features policy convergence *only* in the special case that is the PVM, while generically, there is policy divergence in equilibrium. Also, comparative statics effects (i.e., which primitives influence equilibrium policy choice, and which ones do not) differ substantially between the generalized model and the PVM.

Our results, in particular for the class of models where voters have non-UCR preferences, open several interesting avenues for future research. First, one can focus more closely on particular applications, such as we do in Krasa and Polborn (2010b), where we formalize the notion of issue-ownership, first informally formulated by Petrocik (1996) in the political science literature. Specifically, we consider a setting in which the candidates differ in their ability to produce two public goods (say, ceteris paribus, one candidate has an advantage in supplying national security, while his opponent is better in dealing with the economy) and can propose how to allocate the budget to these two areas. Since the candidates' production levels of the two goods will generally be different even if they propose the same financial budget allocation, it is easy to see that the implied voter preferences violate UCR.

Second, one can analyze the question of candidate selection in more detail. In the present paper, candidates are exogenously endowed with certain fixed characteristics. It may be interesting to add a prior stage to the game where candidates are chosen by parties and their members from two, possibly distinct, sets of available candidates. Interesting questions include how party members, who arguably are primarily interested in policy outcomes rather than in winning per se, choose among potential candidates knowing that these candidates will then go on and choose a policy for the general election in a way to maximize their respective probability of winning.

Appendix A

Proof of Theorem 1. If $a_0 = a_1$, then $c_0 = c_1$ and reflexivity of preferences imply that all voters are indifferent between the candidates. Thus, the winning probabilities as well as votes shares are 0.5. Let (a_0, a_1) be a Nash equilibrium. If Candidate j's payoff were strictly less than 0.5 in this equilibrium, then Candidate j could increase the payoff to 0.5 by using the same policy as the other agent. However, since $W^1(\omega, a_0, a_1) = 1 - W^0(\omega, a_0, a_1)$ this implies $\int W^j(\omega, a_0, a_1)\, d\mu(\omega) = 0.5$, i.e., in equilibrium (a_0, a_1) each candidate's winning probability is 0.5. The same argument holds for the vote shares (simply replace W by V).

We now prove that (a_1, a_1) is Nash equilibrium. Suppose by way of contradiction that there exists a deviation $\tilde{a}_i$ that makes Candidate i strictly better off. If $i = 0$ then Candidate 0 would have used $\tilde{a}_0$ against a_1 thereby increasing his payoff, resulting in a winning probability that is strictly greater than 0.5. This contradicts the assumption that (a_0, a_1) is a Nash equilibrium (as the candidates' winning probability in (a_0, a_1) is 0.5). Thus, we can assume that $i = 1$, i.e., $\tilde{a}_1$ played against a_1 results in an ex-ante winning probability that is strictly greater than 0.5. However, $c_0 = c_1$ implies that $W^0(\omega, a_0, a_1) = W^1(\omega, a_1, a_0)$. Thus, $0.5 < \int W^1(\omega, a_1, \tilde{a}_1)\, d\mu(\omega) = \int W^0(\omega, \tilde{a}_1, a_1)\, d\mu(\omega) \leqslant 0.5$, where the last inequality follows since (a_0, a_1) is a Nash equilibrium with winning probabilities 0.5. This contradiction proves that (a_1, a_1) is a Nash equilibrium. Similarly, it follows that (a_0, a_0) is a Nash equilibrium. Again, the same argument applies to votes share maximization.

Now suppose that (a_0, a_1) is a strict Nash equilibrium. If $a_0 \neq a_1$ then the previous argument implies that (a_0, a_0) is also a Nash equilibrium resulting in the same winning probability, which contradicts the assumption that (a_0, a_1) is strict. Thus, $a_0 = a_1 = \bar{a}$. Suppose by way of contradiction that there exists another pure strategy Nash equilibrium (a', a'), where $a' \neq \bar{a}$

(because of the first part of the proof we can assume that both candidates use the same strategy). Since the equilibrium $(\bar{a}, \bar{a})$ is strict we get $0.5 = \int W^0(\omega, \bar{a}, \bar{a})\, d\mu(\omega) > \int W^0(\omega, a', \bar{a})\, d\mu(\omega)$. Thus, $W^0 + W^1 = 1$ implies $\int W^1(\omega, a', \bar{a})\, d\mu(\omega) > 0.5$. Hence, (a', a') is not a Nash equilibrium since there exists a profitable deviation for Candidate 1, a contradiction. The same contradiction obtains for vote share maximization.

Finally, suppose that there exists a mixed strategy equilibrium. Without loss of generality we can assume that Candidate 0 mixes with strictly positive probability. The prove also works the same way of vote-share maximization. The argument in the previous paragraph implies that $\int W^1(\omega, a, \bar{a})\, d\mu(\omega) \geqslant 0.5$ for all $a \in A$, and that the inequality is strict for $a \neq \bar{a}$. Similarly, $\int W^0(\omega, \bar{a}, a)\, d\mu(\omega) \geqslant 0.5$ for all $a \in A$. The first inequality and the fact that Candidate 0 mixes imply that by choosing $a_1 = \bar{a}$ with probability 1, Candidate 1 gets a winning probability that is strictly greater than 0.5. The second inequality implies that Candidate 0's winning probability must be at least 0.5. Thus, the winning probabilities add to a number strictly greater than 1, a contradiction. Hence, there does not exist a mixed strategy equilibrium. □

Proof of Theorem 2. We start by proving that statement 2 implies statement 1. Since f and g are continuous, the implied preferences are continuous. In remains to prove that UCR holds. Let $(c, b) \succcurlyeq (c', b)$. Then $g(f(c), b) \geqslant g(f(c'), b)$. Since g is strictly monotone in the first argument this implies $f(c) \geqslant f(c')$. Again, strict monotonicity implies $g(f(c), b') \geqslant g(f(c'), b')$, which implies $(c, b') \succcurlyeq (c', b')$, i.e., UCR holds.

We now prove that statement 1 implies statement 2. Define preferences $\succcurlyeq^C$ on C as follows: $c \succcurlyeq^C c'$ if there exists $a \in A$ with $(c, a) \succcurlyeq (c', a)$. Note that these preferences are well defined. In particular, the ability to uniformly rank candidates in state ω implies that $(c, a') \succcurlyeq (c', a')$ for any $a' \in A$. Further preferences $\succcurlyeq^C$ are complete since $\succcurlyeq$ are complete and therefore either $(c, a) \succcurlyeq (c', a)$ or $(c', a) \succcurlyeq (c, a)$ must be satisfied. In the first case $c \succcurlyeq^C c'$ while in the second case $c' \succcurlyeq^C c$. Transitivity of $\succcurlyeq^C$ follows also immediately from transitivity of $\succcurlyeq$. In particular, suppose that $c \succcurlyeq^C c'$ and $c' \succcurlyeq^C c''$. Then for any $a \in A$ we get $(c, a) \succcurlyeq (c', a)$ and $(c', a) \succcurlyeq (c'', a)$. Thus, $(c, a) \succcurlyeq (c'', a)$, which implies that $c \succcurlyeq^C c''$.

Since C is a separable metric space and since preferences are continuous, there exists a continuous utility function f that describes preferences $\succcurlyeq^C$, i.e., $f(c) \geqslant f(c')$ if and only if $c \succcurlyeq^C c'$. Let $Y = f(C)$ and $c, c' \in f^{-1}(y)$ for some $y \in Y$. We now define preferences on $Y \times A$ as follows: $(y, a) \succcurlyeq' (y', a')$ if and only if there exist $c \in f^{-1}(y)$ and $c' \in f^{-1}(y')$ with $(c, a) \succcurlyeq (c', a')$.

To show that these preferences are well defined, let $\hat{c} \in f^{-1}(y)$ and $\hat{c}' \in f^{-1}(y')$. We must show that $(\hat{c}, a) \succcurlyeq (\hat{c}', a')$. $f(c) = f(\hat{c})$ and $f(c') = f(\hat{c}')$ and the fact that f is a utility function for $\succcurlyeq^C$ implies that $(c, a) \sim (\hat{c}, a)$ and $(c', a') \sim (\hat{c}', a')$. Thus, $(\hat{c}, a) \sim (c, a) \succcurlyeq (c', a') \sim (\hat{c}', a')$.

Completeness of preferences $\succcurlyeq'$ follows immediately from completeness of $\succcurlyeq$. To prove transitivity, let $(y, a) \succcurlyeq' (y', a')$ and $(y', a) \succcurlyeq' (y'', a')$. This implies $(c, a) \succcurlyeq (c', a')$ and $(\hat{c}', a') \succcurlyeq (c'', a'')$, where $c \in f^{-1}(y)$, $c', \hat{c}' \in f^{-1}(y')$ and $c'' \in f^{-1}(y'')$. Since $c', \hat{c}' \in f^{-1}(y')$ we get $(c', a') \sim (\hat{c}', a')$. Thus, transitivity of $\succcurlyeq$ implies $(c, a) \succcurlyeq (c'', a'')$, and therefore $(y, a) \succcurlyeq' (y'', a'')$.

Next, we show continuity of $\succcurlyeq'$. Let (y_i, a_i), $i \in \mathbb{N}$ be a sequence with limit (y, a), and let $(\bar{y}, \bar{a}) \in Y \times A$, such that $(y_i, a_i) \succcurlyeq' (\bar{y}, \bar{a})$ for all $i \in \mathbb{N}$. We must show that $(y, a) \succcurlyeq' (\bar{y}, \bar{a})$.

For each $i \in \mathbb{N}$ let $c_i \in f^{-1}(y_i)$. Since C is compact, there exists a subsequence c_{i_k}, $k \in \mathbb{N}$ that converges. Let $c = \lim_{k \to \infty} c_{i_k}$. Continuity of f implies $f(c) = \lim_{k \to \infty} f(c_{i_k}) = \lim_{k \to \infty} y_{i_k} = y$. Since $(y_{i_k}, a_{i_k}) \succcurlyeq' (\bar{y}, \bar{a})$ if follows that $(c_{i_k}, a_{i_k}) \succcurlyeq (\bar{c}, \bar{a})$ for some $\bar{c} \in f^{-1}(\bar{y})$. Continuity of preferences $\succcurlyeq$ implies that $(c, a) \succcurlyeq (\bar{c}, \bar{a})$. Hence $(y, a) \succcurlyeq' (\bar{y}, \bar{a})$.

Similarly, it follows that if $(y_i, a_i) \preccurlyeq' (\bar{y}, \bar{a})$ for all $i \in \mathbb{N}$ then $(y, a) \preccurlyeq' (\bar{y}, \bar{a})$. Thus, preferences $\succcurlyeq'$ are continuous.

Next, note that preferences $\succcurlyeq'$ are strictly monotone in y. In particular, let (y, a), $(y', a) \in Y \times A$ with $y > y'$. Let $c \in f^{-1}(y)$ and $c' \in f^{-1}(y')$. Because f is a utility function describing preferences on C it follows that $c \succ^C c'$. This, in turn implies $(c, a) \succ (c', a)$, and therefore $(y, a) \succ' (y', a)$.

Because $Y \times A$ is again a separable metric space, and the preferences $\succcurlyeq'$ on $Y \times A$ are continuous, there exists a utility function g that describes preferences $\succcurlyeq'$. Strict monotonicity of preferences in y implies that g is strictly monotone in y. Finally, $u(a) = g(f(c), a)$ is a continuous utility function that describes preferences $\succcurlyeq$. □

Proof of Theorem 3. Note that if preferences are UCR then $(c_0, a) \succcurlyeq^\ell_\omega (c_1, a)$ if and only if $(c_0, a') \succcurlyeq^\ell_\omega (c_1, a')$ for any citizen ℓ and for any state $\omega \in \Omega$. Thus, citizens' voting behavior is the same if both candidates choose a or if both choose a'. Thus, the winning probabilities as well as vote shares do not change for candidates $j = 1, 2$, i.e.,

$$W^j(\omega, a, a) = W^j(\omega, a', a') \quad \text{and} \quad V^j(\omega, a, a) = V^j(\omega, a', a'), \quad \text{for all } a, a' \in A. \tag{30}$$

We prove the result for the case where candidates maximize the winning probability. To get the prove for expected vote-share maximizing one only needs to replace $W^i(\cdot)$ by $V^i(\cdot)$.

Suppose by way of contradiction that there exists a strict Nash equilibrium (a_0, a_1) with $a_0 \neq a_1$. Then

$$\int W^0(\omega, a_0, a_1)\, d\mu(\omega) > \int W^0(\omega, a_1, a_1)\, d\mu(\omega), \tag{31}$$

$$\int W^1(\omega, a_0, a_1)\, d\mu(\omega) > \int W^1(\omega, a_0, a_0)\, d\mu(\omega). \tag{32}$$

Table 2
Possible cases for non-UCR preferences.

(a) Case 1			(b) Case 2			(c) Case 3			(d) Case 4		
	(c_1,a)	(c_1,a')		(c_1,a)	(c_1,a')		(c_1,a)	(c_1,a')		(c_1,a)	(c_1,a')
(c_0,a)	1, 0	0, 1	(c_0,a)	1, 0	1, 0	(c_0,a)	1, 0	1, 0	(c_0,a)	1, 0	0, 1
(c_0,a')	1, 0	0, 1	(c_0,a')	1, 0	0, 1	(c_0,a')	0, 1	0, 1	(c_0,a')	0, 1	0, 1

Eqs. (31), (30), and the fact that $W^0 + W^1 = 1$ imply

$$\int W^1(\omega, a_0, a_1)\, d\mu(\omega) < \int W^1(\omega, a_1, a_1)\, d\mu(\omega) = \int W^1(\omega, a_0, a_0)\, d\mu(\omega). \tag{33}$$

But (33) contradicts (32). Thus, in any strict Nash equilibrium $a_0 = a_1 = a$.

Next, we prove uniqueness of the Nash equilibrium (a, a). First, suppose that there exists another pure strategy Nash equilibrium (a_0, a_1). Since the Nash equilibrium (a, a) is strict, it follows that $a_0, a_1 \neq a$. Further, $\int W^1(\omega, a, a)\, d\mu(\omega) > \int W^1(\omega, a, a_1)\, d\mu(\omega)$ and $\int W^0(\omega, a, a)\, d\mu(\omega) > \int W^0(\omega, a_0, a)\, d\mu(\omega)$. Since $W^0 + W^1 = 1$ we get

$$\int W^0(\omega, a, a)\, d\mu(\omega) < \int W^0(\omega, a, a_1)\, d\mu(\omega); \quad \text{and} \tag{34}$$

$$\int W^1(\omega, a, a)\, d\mu(\omega) < \int W^1(\omega, a_0, a)\, d\mu(\omega). \tag{35}$$

Eqs. (34), (35) and the fact that (a_0, a_1) is a Nash equilibrium implies

$$\int W^0(\omega, a, a)\, d\mu(\omega) < \int W^0(\omega, a, a_1)\, d\mu(\omega) \leqslant \int W^0(\omega, a_0, a_1)\, d\mu(\omega); \tag{36}$$

$$\int W^1(\omega, a, a)\, d\mu(\omega) < \int W^1(\omega, a_0, a)\, d\mu(\omega) \leqslant \int W^0(\omega, a_0, a_1)\, d\mu(\omega). \tag{37}$$

Since $W^0 + W^1 = 1$, adding (36) and (37) yields a contradiction. Thus, the Nash equilibrium is unique among all pure strategy equilibria. The remainder of the proof, that there is no mixed strategy equilibrium, is identical to the last step in the proof of Theorem 1. □

Proof of Theorem 4. Since one individual has non-UCR preferences, there exist policies a, a' such that $(c_0, a) \succ u(c_1, a)$ and $(c_0, a') \preccurlyeq (c_1, a')$. If all preferences are strict, we get the cases for the person's voting behavior listed in Table 2.

Without loss of generality assume that $c_1 > c_0$. Let $v: A \to \mathbb{R}$ with $v(a) > v(a') > v(a'')$ for all $a'' \in A \setminus \{a, a'\}$, and $v(a) - v(a') > c_1 - c_0$. Similarly, let $\tilde{v}: A \to \mathbb{R}$ with $\tilde{v}(a') > \tilde{v}(a) > \tilde{v}(a'')$ for all $a'' \in A \setminus \{a, a'\}$, and $\tilde{v}(a') - \tilde{v}(a) > c_1 - c_0$. Consider the following four types of UCR voters, described by their utility functions.

Type $(\boldsymbol{a}, \mathbf{c_0})$: $u(\hat{a}, c) = v(\hat{a}) - |c - c_0|$.
Type $(\boldsymbol{a}, \mathbf{c_1})$: $u(\hat{a}, c) = v(\hat{a}) - |c - c_1|$.
Type $(\boldsymbol{a}', \mathbf{c_0})$: $u(\hat{a}, c) = \tilde{v}(\hat{a}) - |c - c_0|$.
Type $(\boldsymbol{a}', \mathbf{c_1})$: $u(\hat{a}, c) = \tilde{v}(\hat{a}) - |c - c_1|$.

If one candidate proposes a while the other proposes a', then each of these four types votes for the candidate that offers the most preferred policy choice. If one candidate offers a or a' while the other offers a policy $a'' \in A \setminus \{a, a'\}$, then all voters will support the candidate who offers a or a'. Finally, if both candidates propose a or both candidate propose a', then voters will support the candidate according to their fixed characteristic. That is types (a, c_0) and (a', c_0) vote for Candidate 0, while (a, c_1) and (a', c_1) vote for Candidate 1 (note that since no voter abstains maximizing the number of votes is equivalent to maximizing the vote share).

Now consider case 1. Suppose there are two states $\Omega = \{\omega_1, \omega_2\}$ that are equally likely, and two voters other than the non-UCR type. In state ω_1 these two voters are of type (a, c_0) and (a, c_1), respectively. In state ω_2 they are of type (a, c_0) and (a', c_1). This generates exactly the payoffs in Table 1, where a, a' is the strict Nash equilibrium.

In case 2, we change the types to (a', c_0), (a', c_1) in state ω_1, and (a, c_0), (a', c_1) in state ω_2. The resulting payoff matrix is given in Table 3 and a, a' is again the strict Nash equilibrium both when maximizing the winning probability or the vote share. We can use the same types in the two states to generate the expected vote share for case 3 in the table, where (a, a') is again the strict Nash equilibrium. In case 4, we use the same types as in case 1.

Next, consider the game in which candidate maximize the winning probability. We add the same UCR types as in the game with vote share maximization. The resulting payoff matrices are given in Table 4. If follows immediately that (a, a') is the strict Nash equilibrium.

Next, consider the case where $(c_0, a') \sim (c_1, a)$ for the voter with the non-UCR preferences. Then in all four cases we change the probabilities of states ω_1 and ω_2 to 0.6 and 0.4, respectively to get a strict equilibrium. For policies a', a', the

Table 3
Expected votes after UCR voters are added.

(a) Case 2			(b) Case 3			(c) Case 4		
	(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')
(c_0, a)	2, 1	1.5, 1.5	(c_0, a)	2, 1	1.5, 1.5	(c_0, a)	2, 1	1.5, 1.5
(c_0, a')	2.5, 0.5	1, 2	(c_0, a')	1.5, 1.5	1, 2	(c_0, a')	0.5, 2.5	1, 2

Table 4
Payoff matrix for winning probability maximization.

(a) Case 1			(b) Case 2			(c) Case 3			(d) Case 4		
	(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')
(c_0, a)	1, 0	$\frac{1}{2}, \frac{1}{2}$	(c_0, a)	1, 0	$\frac{1}{2}, \frac{1}{2}$	(c_0, a)	1, 0	$\frac{1}{2}, \frac{1}{2}$	(c_0, a)	1, 0	$\frac{1}{2}, \frac{1}{2}$
(c_0, a')	$\frac{1}{2}, \frac{1}{2}$	0, 1	(c_0, a')	1, 0	0, 1	(c_0, a')	$\frac{1}{2}, \frac{1}{2}$	0, 1	(c_0, a')	0, 1	0, 1

Table 5
Expected vote shares when $(c_0, a') \sim (c_1, a')$ for the non-UCR voter.

(b) Case 2			(c) Case 3			(d) Case 4		
	(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')		(c_1, a)	(c_1, a')
(c_0, a)	$\frac{2}{3}, \frac{1}{3}$	$\frac{1.6}{3}, \frac{1.4}{3}$	(c_0, a)	$\frac{2}{3}, \frac{1}{3}$	$\frac{1.6}{3}, \frac{1.4}{3}$	(c_0, a)	$\frac{2}{3}, \frac{1}{3}$	$\frac{1.6}{3}, \frac{1.4}{3}$
(c_0, a')	$\frac{1.4}{3}, \frac{1.6}{3}$	$\frac{1}{2}, \frac{1}{2}$	(c_0, a')	$\frac{2.4}{3}, \frac{0.6}{3}$	$\frac{1}{2}, \frac{1}{2}$	(c_0, a')	$\frac{0.4}{3}, \frac{2.6}{3}$	$\frac{1}{2}, \frac{1}{2}$

non-UCR candidate is indifferent, and therefore abstains. The resulting expected vote shares are provided in Table 5 (case 3 is left out since it is identical to case 1).

In all four cases (a, a') is again the unique Nash equilibrium. In the case of winning probability maximizing (a, a') is again the strict Nash equilibrium. The payoff matrices resemble those in Table 4, except that for strategies (a, a') the winning probabilities are 0.6 and 0.4, respectively. If strategies are (a', a) then in cases 1 and 3 the winning probabilities are 0.4, and 0.6.

Similar constructions also apply if the non-UCR voter is indifferent between (c_0, a) and (c_1, a') or (c_0, a') and (c_1, a). □

A.1. Derivation of equations used in Section 5.2

To solve for the equilibrium we proceed as follows. We first transforming the coordinates using M, which results in indifference curves that are circles. We then rotate the coordinates such that the fixed characteristic, c, is again on the horizontal axis. This can be done by applying the matrix

$$O = \begin{pmatrix} \frac{1}{\sqrt{1+\kappa_1^2\kappa_2^2}} & -\frac{\kappa_1\kappa_2}{\sqrt{1+\kappa_1^2\kappa_2^2}} \\ \frac{\kappa_1\kappa_2}{\sqrt{1+\kappa_1^2\kappa_2^2}} & \frac{1}{\sqrt{1+\kappa_1^2\kappa_2^2}} \end{pmatrix}. \tag{38}$$

Note that

$$O \cdot M \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} c\sqrt{1+\kappa_1^2\kappa_2^2} \\ 0 \end{pmatrix} \quad \text{and} \quad O \cdot M \cdot \begin{pmatrix} 0 \\ a \end{pmatrix} = \frac{a}{\sqrt{1+\kappa_1^2\kappa_2^2}} \begin{pmatrix} \kappa_2(1-\kappa_1^2) \\ \kappa_1(1+\kappa_2^2) \end{pmatrix}. \tag{39}$$

For any c and a let

$$\xi_C(c) = c\sqrt{1+\kappa_1^2\kappa_2^2} \quad \text{and} \quad \xi_A(a) = \frac{a\kappa_1(1+\kappa_2^2)}{\sqrt{1+\kappa_1^2\kappa_2^2}}. \tag{40}$$

Let $\tilde{c}_i = \xi(c_i)$, for Candidates $i = 0, 1$, and $\tilde{a} = \xi_A(a)$. Define β by (14). Then (39) implies that we have a new voting game in which Candidate i can choose policies $(\tilde{c}_i + \beta\tilde{a}, \tilde{a})$, and voters have Euclidean preferences over $(\tilde{c}, \tilde{a})$.

Voter type (δ_j, θ_j) in the original voting game, corresponds to type $(\tilde{\delta}_j + \beta\tilde{\theta}_j, \tilde{\theta}_j)$ in the transformed game, where $\tilde{\delta} = \xi_C(\delta)$ and $\tilde{\theta}_j = \xi_A(\theta_j)$. In the transformed game indifferences curves are circles. Thus, (δ_j, θ_j) preferes Candidate 0 to Candidate 1 if and only if

$$(\tilde{\delta}_j + \beta\tilde{\theta}_j - \tilde{c}_0 - \beta\tilde{a}_0)^2 + (\tilde{\theta}_j - \tilde{a}_0)^2 > (\tilde{\delta}_j + \beta\tilde{\theta}_j - c_1 - \beta\tilde{a}_1)^2 + (\tilde{\theta}_j - \tilde{a}_1)^2. \tag{41}$$

Eq. (41) is equivalent to

$$\tilde{\delta}_j < \frac{1}{2}\left[\frac{(\tilde{c}_1+\beta\tilde{a}_1)^2-(\tilde{c}_0+\beta\tilde{a}_0)^2+\tilde{a}_1^2-\tilde{a}_0^2-2\tilde{\theta}_j(\tilde{a}_1-\tilde{a}_0)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}-2\beta\tilde{\theta}_j\right],$$

which implies

$$\delta_j < \frac{1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[\frac{(\tilde{c}_1+\beta\tilde{a}_1)^2-(\tilde{c}_0+\beta\tilde{a}_0)^2+\tilde{a}_1^2-\tilde{a}_0^2-2\tilde{\theta}_j(\tilde{a}_1-\tilde{a}_0)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}-2\beta\tilde{\theta}_j\right].$$

The optimization problems of Candidates 0 and 1 are therefore

$$\max_{\tilde{a}_0}\sum_{j=1}^{J}\lambda_j F_j\left(\frac{1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[\frac{(\tilde{c}_1+\beta\tilde{a}_1)^2-(\tilde{c}_0+\beta\tilde{a}_0)^2+\tilde{a}_1^2-\tilde{a}_0^2-2\tilde{\theta}_j(\tilde{a}_1-\tilde{a}_0)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}-2\beta\tilde{\theta}_j\right]\right), \tag{42}$$

$$\min_{\tilde{a}_1}\sum_{j=1}^{J}\lambda_j F_j\left(\frac{1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[\frac{(\tilde{c}_1+\beta\tilde{a}_1)^2-(\tilde{c}_0+\beta\tilde{a}_0)^2+\tilde{a}_1^2-\tilde{a}_0^2-2\tilde{\theta}_j(\tilde{a}_1-\tilde{a}_0)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}-2\beta\tilde{\theta}_j\right]\right). \tag{43}$$

Let

$$k(\tilde{a}_0,\tilde{a}_1)=\frac{(\tilde{c}_1+\beta\tilde{a}_1)^2-(\tilde{c}_0+\beta\tilde{a}_0)^2+\tilde{a}_1^2-\tilde{a}_0^2-2\tilde{\theta}_j(\tilde{a}_1-\tilde{a}_0)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)} \quad \text{and} \quad K=\frac{1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}.$$

Then the first-order conditions are

$$\sum_{j=1}^{J}\lambda_j f_j\big(K\big(k(\tilde{a}_0,\tilde{a}_1)-2\beta\tilde{\theta}_j\big)\big)K\left[\frac{-2\beta(\tilde{c}_0+\beta\tilde{a}_0)-2\tilde{a}_0+2\tilde{\theta}_j+k(\tilde{a}_0,\tilde{a}_1)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}\right]=0; \tag{44}$$

$$-\sum_{j=1}^{J}\lambda_j f_j\big(K\big(k(\tilde{a}_0,\tilde{a}_1)-2\beta\tilde{\theta}_j\big)\big)K\left[\frac{-2\beta(\tilde{c}_1+\beta\tilde{a}_1)-2\tilde{a}_1+2\tilde{\theta}_j+k(\tilde{a}_0,\tilde{a}_1)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}\right]=0. \tag{45}$$

Suppose $\tilde{a}_0$ and $\tilde{a}_1$ satisfy condition (15) discussed in the main text. If, in addition $\tilde{a}_0$ and $\tilde{a}_1$ satisfy (44) then $\tilde{a}_0$ and $\tilde{a}_1$ also satisfy (45). Substituting (15) into (44) yields

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{\tilde{c}_0+\tilde{c}_1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\right)\frac{(1+\beta^2)}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[\frac{\beta}{2}+\big(1+\beta^2\big)\frac{\tilde{\theta}_j-\tilde{a}_0}{\tilde{c}_1-\tilde{c}_0}\right]=0, \tag{46}$$

which, using the definition of $\tilde{c}_i$, $\tilde{a}_i$ and $\tilde{\theta}_i$ is equivalent to

$$\sum_{j=1}^{J}\lambda_j f_j\left(\frac{c_0+c_1}{2}\right)\left[\frac{\beta}{2}+\big(1+\beta^2\big)\frac{\xi_A(\theta_j-a_0)}{\xi_C(c_1-c_0)}\right]=0. \tag{47}$$

Continuity of (47) in a_0 immediately implies that there exists a solution. To get a_1 we use the fact that $\tilde{a}_0=\xi_A(a_0)$ and then apply (15) to get $\tilde{a}_1$. Finally, $a_1=\xi_A^{-1}(\tilde{a}_1)$, implies condition (20).

Next, we derive the second-order condition. The derivative of the left-hand side of (44) with respect to $\tilde{a}_0$ is

$$\sum_{j=1}^{J}\lambda_j f_j'\big(K\big(k(\tilde{a}_0,\tilde{a}_1)-2\beta\tilde{\theta}_j\big)\big)K\left(\frac{\partial k(\tilde{a}_0,\tilde{a}_1)}{\partial\tilde{a}_0}\right)^2+\sum_{j=1}^{J}\lambda_j f_j\big(K\big(k(\tilde{a}_0,\tilde{a}_1)-2\beta\tilde{\theta}_j\big)\big)K\left(\frac{\partial^2 k(\tilde{a}_0,\tilde{a}_1)}{\partial^2\tilde{a}_0}\right). \tag{48}$$

Next,

$$\frac{\partial^2 k(a_0,a_1)}{\partial^2 a_0}=-\frac{2(1+\beta^2)}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}+\frac{1+\beta}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}\frac{\partial k(\tilde{a}_0,\tilde{a}_1)}{\partial\tilde{a}_0}. \tag{49}$$

At any critical value of a_0, (44) must be satisfied. Thus,

$$\sum_{j=1}^{J}\lambda_j f_j\big(K\big(k(\tilde{a}_0,\tilde{a}_1)-2\beta\tilde{\theta}_j\big)\big)K\frac{1+\beta}{\tilde{c}_1-\tilde{c}_0+\beta(\tilde{a}_1-\tilde{a}_0)}\frac{\partial k(\tilde{a}_0,\tilde{a}_1)}{\partial\tilde{a}_0}=0. \tag{50}$$

If $\hat{a}_0$ and $\hat{a}_1$ satisfy (15) then $\hat{a}_0 > \hat{a}_1$ and

$$\tilde{c}_1 - \tilde{c}_0 + \beta(\hat{a}_1 - \hat{a}_0) = \frac{1}{1+\beta^2}(\tilde{c}_1 - \tilde{c}_0) > 0.$$

Hence there exists $\underline{a} < \hat{a}_1 < \hat{a}_0 < \bar{a}$ such that $\tilde{c}_1 - \tilde{c}_0 + \beta(\tilde{a}_1 - \tilde{a}_0) > 0$ for any $\tilde{a}_0, \tilde{a}_1 \in [\underline{a}, \bar{a}]$.

As a consequence, (48), (49), and (50) imply that the second-order condition is

$$\sum_{j=1}^{J} \lambda_j K\left[f_j'\big(K\big(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j\big)\big)\left(\frac{\partial k(\tilde{a}_0,\tilde{a}_1)}{\partial \tilde{a}_0}\right)^2 - f_j\big(K\big(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j\big)\big)\frac{2(1+\beta^2)}{\tilde{c}_1 - \tilde{c}_0 + \beta(\tilde{a}_1 - \tilde{a}_0)}\right] < 0.$$

Since $\tilde{c}_1 - \tilde{c}_0 + \beta(\tilde{a}_1 - \tilde{a}_0) > 0$, this is equivalent to

$$\sum_{j=1}^{J} \lambda_j \left[\frac{f_j'(K(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j))}{f_j(K(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j))} \frac{(-2\beta(\tilde{c}_0 + \beta\tilde{a}_0) - 2\tilde{a}_0 + 2\tilde{\theta}_j + k(\tilde{a}_0,\tilde{a}_1))^2}{(2(1+\beta^2))(\tilde{c}_1 - \tilde{c}_0 + \beta(\tilde{a}_1 - \tilde{a}_0))} - 1\right] < 0. \tag{51}$$

Similarly, the second-order condition for (43) is

$$\sum_{j=1}^{J} \lambda_j \left[\frac{f_j'(K(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j))}{f_j(K(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j))} \frac{(-2\beta(\tilde{c}_0 + \beta\tilde{a}_1) - 2\tilde{a}_1 + 2\tilde{\theta}_j + k(\tilde{a}_0,\tilde{a}_1))^2}{(2(1+\beta^2))(\tilde{c}_1 - \tilde{c}_0 + \beta(\tilde{a}_1 - \tilde{a}_0))} - 1\right] < 0. \tag{52}$$

Both second-order conditions are satisfied at the solutions of the first-order conditions. In particular,

$$f_j'\big(K\big(k(\tilde{a}_0,\tilde{a}_1) - 2\beta\tilde{\theta}_j\big)\big) = f_j'\left(\frac{c_0 + c_1}{2}\right) = 0,$$

which implies that the left-hand sides of (51) and (52) are $-\sum_{j=1}^{J} \lambda_j < 0$. Thus, we have a local equilibrium that is strict.

We next show that (a_0, a_1) is characterized by (15) and (47) is the unique equilibrium pure or mixed.

In particular, we change coordinates, by using the orthogonal matrix

$$D = \begin{pmatrix} \frac{1}{\sqrt{1+\beta^2}} & -\frac{\beta}{\sqrt{1+\beta^2}} \\ \frac{\beta}{\sqrt{1+\beta^2}} & \frac{1}{\sqrt{1+\beta^2}} \end{pmatrix}. \tag{53}$$

Because D is a rotation, the indifference curves of voters remain circles. In the previous voting game, policy choices where on lines of the form $(\tilde{c}_i + \beta a, a)$. Now note that after applying D the lines on which policies are chosen are vertical. Next, (15) implies

$$D \cdot \left[\begin{pmatrix} \tilde{c}_1 + \beta\tilde{a}_1 \\ \tilde{a}_1 \end{pmatrix} - \begin{pmatrix} \tilde{c}_0 + \beta\tilde{a}_0 \\ \tilde{a}_0 \end{pmatrix}\right] = D \cdot \begin{pmatrix} \frac{1}{1+\beta^2}(\tilde{c}_1 - \tilde{c}_0) \\ -\frac{\beta}{1+\beta^2}(\tilde{c}_1 - \tilde{c}_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+\beta^2}} \\ 0 \end{pmatrix}.$$

Thus, both candidates choose the same policy in the transformed voting game. Further, the second-order conditions imply that the equilibrium is strict. Since preferences are circles, they are UCR. As a consequence, Theorem 3 implies that the equilibrium in the transformed voting game is unique. Hence $(\tilde{a}_0, \tilde{a}_1)$ is the unique equilibrium in the voting game with fixed positions $\tilde{c}_i$ and feasible policy lines $(\tilde{c}_i + \beta a, a)$.

We next show that the arithmetic mean of the candidates' policies $\bar{a} = (a_0 + a_1)/2$ is independent of κ_1 and κ_2. In particular, using (15) to substitute a_0 for a_1 in (45) yields

$$\sum_{j=1}^{J} \lambda_j f_j\left(\frac{\tilde{c}_0 + \tilde{c}_1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\right)\frac{(1+\beta^2)}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[-\frac{\beta}{2} + \big(1+\beta^2\big)\frac{\tilde{\theta}_j - \tilde{a}_1}{\tilde{c}_1 - \tilde{c}_0}\right] = 0. \tag{54}$$

Adding (46) and (54) yields

$$\sum_{j=1}^{J} \lambda_j f_j\left(\frac{\tilde{c}_0 + \tilde{c}_1}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\right)\frac{(1+\beta^2)^2}{2\sqrt{1+\kappa_1^2\kappa_2^2}}\left[\frac{2\tilde{\theta}_j - (\tilde{a}_0 + \tilde{a}_1)}{\tilde{c}_1 - \tilde{c}_0}\right] = 0. \tag{55}$$

Substituting $\bar{a}$ for $(a_0 + a_1)/2$, applying functions ξ_A and ξ_C, and eliminating constants, (55) simplifies to

$$\sum_{j=1}^{J} \lambda_j f_j\left(\frac{c_0 + c_1}{2}\right)\left[\frac{\theta_j - \bar{a}}{c_1 - c_0}\right] = 0. \tag{56}$$

Thus, the solution $\bar{a}$ of (56) is independent of κ_1 and κ_2.

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