Shanks Workshop

# Ordered Algebras and Logic

February 6–7, 2016

– Booklet of Abstracts –



National Science Foundation, Shanks Endowment, Vanderbilt University,
Consortium for Order Algebra and Logic

## Venue

Department of Mathematics of Vanderbilt University
1326 Stevenson Center
Ground floor, Room 1308

## Organizers

José Gil-Férez, *Vanderbilt University,* USA
Constantine Tsinakis, *Vanderbilt University,* USA

# Schedule

| Saturday, February 6 | |
|---|---|
| 10:00 - 11:00 | Stephen Simpson<br>*Well partial orderings and better partial orderings, with applications to algebra* |
| 11:00 - 12:00 | Nick Galatos<br>*Canonical formulas for $k$-potent commutative residuated lattices* |
| 12:00 - 13:30 | Lunch |
| 13:30 - 14:30 | Riquelmi Cardona<br>*The FEP for some noncommutative varieties of residuated lattices satisfying $x^n = x^m$* |
| 14:30 - 15:30 | Antonio Ledda<br>*New horizons for the Archimedean Property* |
| 15:30 - 16:30 | Sam van Gool<br>*Uniform interpolation and compact congruences* |
| 18:00 - 21:00 | Gathering |

| Sunday, February 7 | |
|---|---|
| 10:00 - 11:00 | Francesco Paoli<br>*An abstract approach to consequence relations* |
| 11:00 - 12:00 | José Gil-Férez<br>*Paraconsistent Weak Kleene and its algebraic counterpart* |
| 12:00 - 13:30 | Lunch |
| 13:30 - 14:30 | James Hart<br>*Prime ideals and the classical topology on graphs* |
| 14:30 - 15:30 | Zack French<br>*The predicate completion of a partial information system* |
| 15:30 - 16:30 | George Metcalfe<br>*Density revisited* |

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# The FEP for some noncommutative varieties of residuated lattices satisfying $x^n = x^m$

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A class of algebras $K$ is said to have the finite embeddability property (FEP) if every finite partial subalgebra of an algebra in $K$ can be embedded into a finite algebra in $K$. We consider varieties of residuated lattices that satisfy $x^n = x^m$ (sometimes this condition is referred to as periodicity) and a balanced noncommutative multiplicative identity.

We consider noncommutative balanced monoid identities that have the property that one of the sides of the identity does not contain a square and prove the FEP for varieties of fully distributive residuated lattices that satisfy a periodic equality and one of these noncommutative identities. This result can be extended to the fully distributive, involutive and cyclic subvarieties.

## The predicate completion of a partial information system

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Originally, partial information systems were introduced as a means of providing a representation of the Smyth powerdomain in terms of order convex substructures of an information-based structure. For every partial information system **S**, there is a new partial information system that is naturally induced by the principal lowersets of the consistency predicate for **S**. In this paper, we show that this new system serves as a completion of the parent system **S** in two ways. First, we demonstrate that the induced system relates to the parent system **S** in much the same way as the ideal completion of the consistency predicate for **S** relates to the consistency predicate itself. Second, we explore the relationship between this induced system and the notion of $D$-completions for posets. In particular, we show that this induced system has a "semi-universal" property in the category of partial information systems coupled with the preorder analog of Scott-continuous maps that is induced by the universal property of the $D$-completion of the principal lowersets of the consistency predicate for the parent system **S**.

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## Canonical formulas for $k$-potent commutative residuated lattices

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Canonical formulas provide useful axiomatizations for varieties and have been considered in the setting of Heyting algebras. We prove that all varieties of $k$-potent commutative residuated lattices can be axiomatized by such formulas. The result makes use of a characterization of the subdirectly irreducible algebras in the variety and also of the local finiteness of the multiplication reducts. Time permitting we

discuss how to extend this result to other varieties of residuated lattices and even to universal classes by replacing canonical formulas with canonical rules.

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## Paraconsistent Weak Kleene and its algebraic counterpart

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(Joint work with S. Bonzio, F. Paoli, and L. Peruzzi)

Paraconsistent Weak Kleene logic (PWK) is the 3-valued logic with two designated values defined through the weak Kleene tables. This is a first attempt to investigate PWK within the perspective and methods of Abstract Algebraic Logic (AAL). We give a Hilbert-style system for PWK and prove a normal form theorem. First, we examine some algebraic structures for PWK, called *involutive bisemilattices*, showing that they are distributive as bisemilattices and that they form a variety, $\mathcal{IBSL}$, generated by the 3-element algebra **WK**; we also prove that every involutive bisemilattice is representable as the Płonka sum over a direct system of Boolean algebras. We then study PWK from the viewpoint of AAL. We show that $\mathcal{IBSL}$ is not the equivalent algebraic semantics of any algebraizable logic and that PWK is neither protoalgebraic nor selfextensional. We fully characterize the deductive filters of PWK on members of $\mathcal{IBSL}$ and the reduced matrix models of PWK. Finally, we investigate PWK with the methods of second-order AAL –we describe the algebra reducts of full basic generalized matrix models of PWK, showing that they coincide with the quasivariety generated by **WK** and explicitly providing a quasiequational basis for it.

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## Prime ideals and the classical topology on graphs

James B. Hart

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(Joint work with B. P. Frazier)

In his 2005 dissertation, Antoine Vella introduced a novel topological structure on graphs and hypergraphs. While this "classical" topology has generated interest among graph theorists, the structure of the open set lattices that arise from it remains unexplored. In this talk, we will introduce the curious structure these lattices possess. For simple graphs with no isolated vertices, we will characterize the prime spectrum of the open set lattice and demonstrate how these graphs may be recaptured from the prime spectrum. Time permitting, we will also discuss ongoing research.

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## New horizons for the Archimedean Property

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(Joint work with F. Paoli and C. Tsinakis)

Although there have been repeated attempts to define concepts of Archimedean algebra for individual classes of residuated lattices, it looks as though there is no all-purpose definition that suits the general case. We suggest as a possible candidate the notion of *locally semisimple* residuated lattice — namely, a residuated lattice whose principal convex subalgebras are semisimple. We characterize the locally semisimple members in the variety of $e$-cyclic residuated lattices, as well as in various other cases of interest. A theorem to the effect that each locally semisimple, prelinear and normal-valued GBL algebra is commutative (subsuming as corollaries several analogous results from the recent literature) is grist to the mill of our proposal's adequacy.

## Density revisited

George Metcalfe
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Proving that the chains of a variety of semilinear residuated lattices embed into dense chains of that variety forms a key step in so-called "standard completeness" proofs for fuzzy logics. In this talk I will revisit the Jenei-Montagna standard completeness proof for the logic MTL and show that their method can be adapted and generalized to obtain embeddings of chains into dense chains for all varieties of integral semilinear residuated lattices axiomatized using equations of a particular form. I will also revisit the Metcalfe-Montagna standard completeness proof for the logic UL, which uses the syntactic elimination of a certain "density" rule for a hypersequent calculus to establish embeddings of chains into dense chains for the variety of commutative semilinear residuated lattices.

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## An abstract approach to consequence relations

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(Joint work with P. Cintula and J. Gil-Férez)

We generalise the Blok-Jonsson account of consequence relations [1], later developed by Galatos and Tsinakis [2], in such a way as to accommodate multiset consequence and some forms of graded consequence. While Blok and Jonsson admit, in place of the sheer formulas of the Tarskian approach, a wider range of formal objects to be manipulated in deductions (including formulas, sequents, equations), these objects are invariably aggregated via set-theoretical union. Our approach, on the other hand, is more general in that different forms of premiss and conclusion aggregation, including multiset sum and various T-conorms, are considered. In its abstract form, thus, a consequence relation is nothing else than a partially ordered monoid with an additional compatible preordering that refines the given partial

ordering, and equivalence of consequence relations is defined via certain po-monoid isomorphisms. Our suggestion has both pros and cons when compared to the proposal in [1]. Its main disadvantage is that a satisfactory account of structurality is beyond the state of the art, whence no analogue of the fascinating theory of substitutions as actions developed in [1] or [2] is available at the moment. The advantage is that a generalisation of matrix semantics is possible within our approach.

**References**


[1] Blok W.J., Jonsson B., *Equivalence of consequence operations*, Studia Logica, 83, 1-3, 2006, pp. 91-110.

[2] Galatos N., Tsinakis C., *Equivalence of consequence relations: An order-theoretic and categorical perspective*, J. Symbolic Logic, 74, 3, 2009, pp. 780-810.


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## Well partial orderings and better partial orderings, with applications to algebra


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A well partial ordering is a partial ordering with no infinite descending sequences and no infinite antichains. This concept arises frequently in algebra. For example, a standard proof of the Hilbert Basis Theorem uses Dickson's Lemma: for each positive integer $k$, the set of $k$-tuples of natural numbers with the product ordering is a well partial ordering. As another application, there is a theorem of Formanek and Lawrence: the group ring of the infinite symmetric group is Noetherian, i.e., it has no infinite ascending chain of two-sided ideals. The class of well partial orderings has certain finitary closure properties. The better partial orderings are a subclass of the well partial orderings, but with analogous infinitary closure properties. The concept of better partial orderings was introduced by Nash-Williams and used by Laver to prove a conjecture of Fraisse: the class of countable linear orderings is well partially ordered under embeddability. As an algebraic application of better partial orderings, we now prove that the group ring of the infinite symmetric group has the antichain condition, i.e., it has no infinite antichain of two-sided ideals.

## Uniform interpolation and compact congruences

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(Joint work with G. Metcalfe and C. Tsinakis)

Starting from the uniform interpolation theorem for intuitionistic logic by A. M. Pitts (1992), we investigate uniform interpolation in a universal algebraic setting. Uniform interpolation for a logic may be viewed as a weaker form of quantifier elimination. This idea is exploited in the 2002 monograph of Ghilardi and Zawadowski to show that under certain conditions, uniform interpolation for a logic implies the existence of a model completion for a corresponding variety (equational class) of algebras. Following the category-theoretic work by Ghilardi and Zawadowski, we obtain algebraic characterizations of the property of existence of left and right uniform interpolants involving the compact congruence lattices of these algebras. Moreover, we identify, among varieties of algebras corresponding to substructural and many-valued logics, several varieties that admit and do not admit these properties.

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