RESEARCH STATEMENT, EXTENDED VERSION

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My interests lie in fields of Harmonic Analysis, Partially Differential Equations, Probability and Statistics. For my research, I use methods from geometric measure theory, potential theory, probability theory, and operator theory. More specifically, my current research interests can be divided into following categories:

- Various methods of distributing points over compact sets; separation and covering properties of these distributions.
- Asymptotic behavior of discrete energy and Chebyshev constants on compact sets; relation to problems of best covering and best packing.
- Boundedness of singular integral operators, such as Hilbert or Cauchy Transforms and Toeplitz operator on Bergman space.

In all of these topics I have made some progress and there are many open problems that I want to solve in the future. In what follows, I explain these problems, their history, difficulties and my motivation for solving them. I also state results that I have already proved and their possible extensions.

1. Distributing points over compact sets

Let $A \subset \mathbb{R}^p$ be a compact set, and $\omega_N = \{x_1, x_2, \dots, x_N\}$ be a *multiset*; i.e., a set of N points counting multiplicities. We define the *separation constant of* ω_N by

$$\delta(\omega_N) := \min_{i\neq j} |x_i - x_j|, \tag{1}$$

and the *covering radius of* ω_N *with respect to* A by

$$\rho_A(\omega_N) := \max_{y\in A} \min_{j} |y - x_j|. \tag{2}$$

We notice that ω_N is always an $\rho_A(\omega_N)$-net in A; i.e., for any point $y \in A$ there is a point $x \in \omega_N$ with $|y - x| \leqslant \rho_A(\omega_N)$. In compressed and 1-bit sensing [34] and [6] the existence of an ε-net with sufficiently small cardinality is important; in my works, I prove that certain **constructive** point distributions can be used as these ε-nets.

The notion of covering radius also plays an important role in various numerical approximation schemes (see, e.g., [28] and [14]).

We introduce a class of sets that arises in various contexts and originates from geometric measure theory.

Definition 1. A compact set $A \subset \mathbb{R}^p$ is called *d-regular*, $d \leqslant p$, if for some positive constants c and C, any $x \in A$ and any positive r with $r < \operatorname{diam}(A)$, we have

$$cr^d \leqslant \mathscr{H}_d(A \cap B(x,r)) \leqslant Cr^d. \tag{3}$$

Here $B(x,r) = \{y \in \mathbb{R}^p : |y - x| < r\}$ and $\mathscr{H}_d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^p$, normalized by $\mathscr{H}_d([0,1]^d) = 1$.

Notice that if A is d-regular, then for some positive constants c_1, c_2 we have for any N-point set $\omega_N \subset A$:

$$\delta(\omega_N) \leqslant c_1 N^{-1/d}, \quad \rho_A(\omega_N) \geqslant c_2 N^{-1/d}. \tag{4}$$

Thus, we call $N^{-1/d}$ an *optimal order of separation and covering* for a d-regular set A. We would like to study the order of separation and covering of point sets ω_N distributed over A according to some specific law.

1.1. **Independent random distributions.** Fix a Borel measure μ supported on A, and let $x_1, \dots, x_N$ be random points independently distributed with respect to μ; i.e., for a Borel set $B \subset A$ let

$$\mathbb{P}(x_j \in B) = \mu(B)/\mu(A).$$

If A is a d-regular set, one can consider $\mathrm{d}\mu = \mathbb{1}_A \mathrm{d}\mathscr{H}_d$. In this case E. Saff and I proved the following asymptotic behavior of expected covering radius; for $A = \mathbb{S}^d$ it proves the formula conjectured in [11].

Theorem 1.1. *Let $A \subset \mathbb{R}^p$ be a d-regular compact set and $\omega_N = \{x_1, \dots, x_N\}$ be a set of N points randomly and independently distributed over A with respect to $\mathbb{1}_A \mathrm{d}\mathscr{H}_d$. Then there exist positive constants c_1 and c_2, such that for any $N \geqslant 1$*

$$c_1 \left(\frac{\log N}{N}\right)^{1/d} \leqslant \mathbb{E}\rho_A(\omega_N) \leqslant c_2 \left(\frac{\log N}{N}\right)^{1/d}. \tag{5}$$

Moreover, if A is a d-dimensional closed $C^{1,1}$–smooth manifold, then

$$\lim_{N\to\infty} \mathbb{E}\rho_A(\omega_N) \cdot \left[\frac{N}{\log N}\right]^{1/d} = \left(\frac{\mathscr{H}_d(A)}{\upsilon_d}\right)^{1/d}, \tag{6}$$

where $\upsilon_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}$ is the volume of a d-dimensional unit ball in $\mathbb{R}^d$.

Further, for the d-dimensional unit ball $\mathbb{B}_d$ we have

$$\lim_{N\to\infty} \mathbb{E}\rho_{\mathbb{B}_d}(\omega_N) \cdot \left[\frac{N}{\log N}\right]^{1/d} = \left(\frac{2(d-1)}{d}\right)^{1/d}, \tag{7}$$

and for the d-dimensional unit cube $[0,1]^d$ we have

$$\lim_{N\to\infty} \mathbb{E}\rho_{[0,1]^d}(\omega_N) \cdot \left[\frac{N}{\log N}\right]^{1/d} = \left(\frac{2^{d-1}}{d\upsilon_d}\right)^{1/d}. \tag{8}$$

We see that in all cases the order of expected covering radius of ω_N is not optimal; however, it differs from the optimal one by a power of $\log N$. Thus, the covering radius $\rho_A(\omega_N)$ is *much more forgiving* than the separation $\delta(\omega_N)$ [1], where together with J. Brauchart et. al. I proved (cf. [11]) that for $A = \mathbb{S}^d$ we have

$$\lim_{N\to\infty} \mathbb{E}\delta(\omega_N) \cdot N^{2/d} = C_d,$$

where C_d is an explicit constant.

[1] Citation from a preprint by Bourgain, Sarnak and Rudnick, [10]

1.2. **Determinantal point processes.** For the sake of simplicity we work only on the unit sphere $\mathbb{S}^d$. Besides considering N independent random points, there is another way to generate N random (but dependent) points on the sphere. Namely, fix a symmetric function $K\colon \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{R}$, such that the integral operator

$$f \mapsto \int_{\mathbb{S}^d} K(x,y) f(y) \mathrm{d}\mathscr{H}_d(y), \;\; f \in L^2(\mathbb{S}^d, \mathscr{H}_d)$$

is a projection on $L^2(\mathbb{S}^d, \mathscr{H}_d)$ with eigenvalues 0 and 1 and trace N. We define N random points $X_1, \ldots, X_N$ on $\mathbb{S}^d$ by their *joint intensities*; i.e., for every $k = 1, \ldots, N$ we set

$$\rho_k(x_1, \ldots, x_k) := \det(K(x_i, x_j))_{i,j=1}^k$$

and require the equality

$$\mathbb{E} \sum_{X_1,\ldots,X_k \text{ distinct}} f(X_1, \ldots, X_k) = \int_{(\mathbb{S}^d)^k} f(x_1, \ldots, x_k) \rho_k(x_1, \ldots, x_k) \mathrm{d}\mathscr{H}_d(x_1) \ldots \mathrm{d}\mathscr{H}_d(x_k)$$

to hold for any bounded measurable symmetric function f on $(\mathbb{S}^d)^k$.

Determinantal point processes attract many attention; in particular, they arise in studies of Coulomb gases [26], 1-componend plasma in 2D [4] and Monte-Carlo methods [3]. A determinantal point process on $\mathbb{S}^2$ also arises from stereographic projection of eigenvalues of certain random matrices, see [1] and [17].

It has been noticed in [1] and [5] that for an N-point determinantal process $\omega_N = \{X_1, \ldots, X_N\}$ with appropriate kernel K, the expected separation $\mathbb{E}\delta(\omega_N)$ has better order than for independently distributed random points. In [1] is is also shown that for the *harmonic ensemble* on $\mathbb{S}^2$ the order of covering radius is better than for independently distributed random points. However, the proof strongly relies on the results about eigenvalues of random matrices.

Problem 1. *I would like to compute the expected covering radius of an N-point determinantal process on* $\mathbb{S}^d$.

I anticipate this problem to require careful study of orthogonal polynomials and their asymptotic behavior.

1.3. **Optimal configurations for energy and polarization.** Electrons restricted to a sphere $\mathbb{S}^2$ repel each other according to the Coulomb law (or Coulomb potential $1/r$), and arrange themselves in order to minimize energy. Atoms arrange themselves to minimize a Hamiltonian; at zero temperature, it reduces to a ground state of minimal energy. The potentials that define corresponding energies often have singularities on the diagonal. The study of "equilibrium states", i.e., arrangements that minimize the energy, helps to explain many observed phenomena; in particular, why atoms sometimes arrange themselves in periodic order. For example, atoms in a snowflake are arranged in a hexagonal lattice.

Precisely, for a multiset $\omega_N = \{x_1, \ldots, x_N\} \subset A$, and a positive number s define the *Riesz s-energy of* ω_N by

$$E_s(\omega_N) := \sum_{i \neq j} \frac{1}{|x_i - x_j|^s}, \tag{9}$$

and the *s-polarization of* ω_N by

$$P_s(A; \omega_N) := \min_y \sum_{j=1}^N \frac{1}{|y - x_j|^s}. \tag{10}$$

Further, define the *optimal N-point s-energy* of A by

$$\mathscr{E}_s(A;N) := \min_{\omega_N \subset A} E_s(\omega_N), \tag{11}$$

and the *optimal N-point s-polarization (or Chebyshev) constant* of A by

$$\mathscr{P}_s(A;N) := \max_{\omega_N \subset A} P_s(A;\omega_N). \tag{12}$$

One possible question to ask is the following: if many electrons on a sphere repel each other according to some potential, how uniformly do they cover the sphere? It is known, see [23] and [16], that if A is d-regular and $s > d$, then for any optimal N-point configuration ω_N for $\mathscr{E}_s(A;N)$, both $\delta(\omega_N)$ and $\rho_A(\omega_N)$ have the optimal order. Together with E. Saff and A. Volberg, I proved the following theorem about optimal configurations for $\mathscr{P}_s(A;N)$.

Theorem 1.2. *Assume A is a convex compact set in $\mathbb{R}^d$ with non-empty interior, $s > d$, and the boundary ∂A is C^2-smooth with non-degenerate Gaussian curvature* [2]*. Then there exists a positive constant c, such that for any $N \geqslant 1$ and any optimal configuration ω_N for $\mathscr{P}_s(A;N)$ we have*

$$\rho_A(\omega_N) \leqslant cN^{-1/d}. \tag{13}$$

The same is true for the unit sphere $\mathbb{S}^d$ and for any spherical cap $A \subset \mathbb{S}^d$. Moreover, the same is true for the unit cube $[0,1]^d$ if $s \geqslant 3d-4$.

In the proof we utilize the fact that the function $x \mapsto |x-y|^{-s}$, $s > d$ is subharmonic; i.e., for every $y \in \mathbb{R}^d$ we have

$$\Delta_x \left(|x-y|^{-s}\right) \geqslant 0.$$

We use it to get rid of the second-order term in Taylor expansion of $f_y(x) = |x-y|^{-s}$; however, the computation works if the point $x \in A$ is away from the boundary ∂A and therefore we can move it in any direction and stay inside A. We use our assumptions on ∂A to show that if $x \in \omega_N$, the optimal configuration for $\mathscr{P}_s(A;N)$, then x can not be too close to ∂A. This approach does not work for the cube $[0,1]^d$, where we can not prove that points from ω_N can not be exactly on the boundary $\partial\left([0,1]^d\right)$. A slight modification of the argument works only under the assumption $s \geqslant 3d-4$, which should be omitted in the future.

Problem 2. *Relax conditions on the set A in Theorem 1.2; in particular, for $A = [0,1]^d$ prove it for any $s > d$.*

In the case $d-2 < s < d$ and $A = \mathbb{S}^d$, it is known that optimal configurations for $\mathscr{E}_s(\mathbb{S}^d;N)$ give the optimal order of separation, see [24] and [13]. Moreover, if $s = d-1$ it is known that these configurations give the optimal order of covering [12]. The approach in [12] requires careful study of Laplacian operator Δ and it's Green function G_Ω in a certain general domain Ω. It is often used that this operator is local; i.e., $\Delta u(x)$ can be defined only by behavior of function u in a neighborhood of x. This allows to use the integration by parts (or the Green) formula. Another key fact is that $|x-y|^{1-d}$ is the fundamental solution for the Laplace equation; i.e., for some constant c_d we have

$$\Delta_x \left(\frac{1}{|x-y|^{d-1}} \right) = c_d \delta_y, \;\; x,y \in \mathbb{R}^{d+1}.$$

[2] Such conditions appear in many problems in harmonic analysis, see, e.g., [21]

Problem 3. *I want to follow Dahlberg's approach [12] to show that optimal configurations for $\mathscr{E}_s(\mathbb{S}^d;N)$ give the optimal order of covering for $d-1<s<d$.*

The difficulty in this problem is that one needs to consider the operator Δ^α for some power α. However, either we define it as a global operator (using Fourier transform) and then the integration by parts formula in a domain will not work; or, if we define it with respect to concrete domain (using the spectral theorem), its fundamental solution will be the Green function for this domain, and not $|x-y|^{-s}$.

2. Discrete energy and polarization on compact sets for non-integrable Riesz kernels

Suppose $A\subset\mathbb{R}^p$ is an image of a compact subset of $\mathbb{R}^d$ via a Lipschitz map, $d\leqslant p$, and $s>d$. Notice that the set A has zero s-Riesz capacity. Then the asymptotic behavior of discrete N-point energy on A, defined in (11), is the following (see [8]):

$$\lim_{N\to\infty}\frac{\mathscr{E}_s(A;N)}{N^{1+s/d}}=\frac{C_{s,d}}{\mathscr{H}_d(A)^{s/d}},$$

where $C_{s,d}$ is a finite constant that depends only on s and d, and not on A. Moreover, if $\omega_N=\{x_1^N,\dots,x_N^N\}$ is an optimal N-point configuration, then

$$\frac{1}{N}\sum_{j=1}^N\delta_{x_j^N}\xrightarrow{*}\frac{\mathscr{H}_d(\cdot\cap A)}{\mathscr{H}_d(A)},\ \ N\to\infty,$$

where the limit $\xrightarrow{*}$ is understood in the weak* sense; i.e., for any set $B\subset A$ with $\mathscr{H}_d(\partial_A B)=0$ we have

$$\frac{\#(\omega_N\cap N)}{N}\to\frac{\mathscr{H}_d(B)}{\mathscr{H}_d(A)},\text{ as }N\to\infty.$$

Notice that no assumptions on the boundary ∂A are necessary; in particular, the theorem holds for a "fat" Cantor set $\mathscr{C}\subset[0,1]$ of positive 1-Lebesgue measure.

Together with S. Borodachov, D. Hardin and E. Saff, I proved the following weaker statement for discrete polarization, defined in (12).

Theorem 2.1. *If $A\subset\mathbb{R}^p$ is a d-dimensional C^1-smooth manifold with $\mathscr{H}_d(\partial A)=0$, and $s>d$, then*

$$\lim_{N\to\infty}\frac{\mathscr{P}_s(A;N)}{N^{s/d}}=\frac{\sigma_{s,d}}{\mathscr{H}_d(A)^{s/d}},\tag{14}$$

where $\sigma_{s,d}$ is a finite constant that depends only on s and d, and not on A. Moreover, if $\omega_N=\{x_1^N,\dots,x_N^N\}$ is an optimal N-point configuration, then

$$\frac{1}{N}\sum_{j=1}^N\delta_{x_j^N}\xrightarrow{*}\frac{\mathscr{H}_d(\cdot\cap A)}{\mathscr{H}_d(A)},\ \ N\to\infty.$$

Problem 4. *Get rid of the smoothness assumption, as well as of $\mathscr{H}_d(\partial A)=0$.*

Let us discuss some difficulties of this problem. If $A\subset\mathbb{R}^d$ has positive d-Lebesgue measure, then proof of Theorem 2.1 goes through approximating A by small cubes from inside. If $\mathscr{H}_d(\partial A)>0$, in particular, if A is a "fat" Cantor set, then such an approximation is not possible; for example, a "fat" Cantor set does not contain any cube.

For the energy, this difficulty results in careful approximation of the general set A by more regular sets A_ε. One particular thing used is monotonicity property; i.e., if $A\subset A_\varepsilon$, then $\mathscr{E}_s(A;N) \geqslant \mathscr{E}_s(A_\varepsilon;N)$ for any positive integer N. Thus, if we can derive an estimate from below for $\mathscr{E}_s(A_\varepsilon;N)$, then we immediately get an estimate from below for $\mathscr{E}_s(A;N)$. For polarization $\mathscr{P}_s$ such a monotonicity does not hold. Thus, some new ideas are required to tackle this problem. I propose to start with an explicit "fat" Cantor set $\mathscr{C}$ and prove that

$$\lim_{N\to\infty} \frac{\mathscr{P}_s(\mathscr{C};N)}{N^s} \to \sigma_{s,1}, \;\; s>1.$$

The next problem deals with precise values for constants $C_{s,2}$ and $\sigma_{s,2}$.

Problem 5. *Prove that for $s>2$*

$$C_{s,2} = \left(\frac{\sqrt{3}}{2}\right)^{s/2} \zeta_\Lambda(s), \tag{15}$$

where $\zeta_\Lambda(s) = \sum_{x\in\Lambda\setminus\{0\}} |x|^{-s}$, and $\Lambda = \{(n+m/2, m\sqrt{3}/2)\colon n,m\in\mathbb{Z}\}$ is the hexagonal lattice in $\mathbb{R}^2$. Further, prove that

$$\sigma_{s,2} = \left(\frac{\sqrt{3}}{2}\right)^{s/2} \cdot \frac{3^{s/2}-1}{2} \zeta_\Lambda(s). \tag{16}$$

Conjectured formula (15) is, in particular, confirmed by the known asymptotic behavior of $C_{s,d}$ as $s\to\infty$; namely, in [7] it is proved that

$$\lim_{s\to\infty} (C_{s,d})^{1/s} = \frac{1}{\lim_{N\to\infty}(N^{1/2}\delta_{[0,1]^d}(N))},$$

where

$$\delta_{[0,1]^d}(N) = \max_{\omega_N\subset[0,1]^d} \delta(\omega_N)$$

is the *best-packing constant* for $[0,1]^d$. In dimensions $d=2,3,8,24$ this constant is known; in particular, for $d=2$, in a microscopic sense, it is achieved by the hexagonal lattice, and

$$\lim_{s\to\infty} (C_{s,2})^{1/s} = \frac{\sqrt[4]{12}}{2}. \tag{17}$$

Using Theorem 1.2, I was able to prove the analogous result on $\sigma_{s,d}$.

Theorem 2.2. *If $\sigma_{s,d}$ is the constant in the right-hand side of* (14)*, then*

$$\lim_{s\to\infty} (\sigma_{s,d})^{1/s} = \frac{1}{\lim_{N\to\infty} N^{1/d}\rho_{[0,1]^d}(N)}, \tag{18}$$

where

$$\rho_{[0,1]^d}(N) := \inf_{\omega_N\subset[0,1]^d} \rho_{[0,1]^d}(\omega_N)$$

is the "best covering constant" for the unit cube $[0,1]^d$. For $d=2$ it is known [22] that $\rho_{[0,1]^2}(N) = \sqrt[4]{12}/3$; thus,

$$\lim_{s\to\infty} (\sigma_{s,2})^{1/s} = \frac{3}{\sqrt[4]{12}}. \tag{19}$$

We also remark that the estimate

$$C_{s,2} \leqslant \left(\frac{\sqrt{3}}{2}\right)^{s/2} \zeta_\Lambda(s),$$

is known [23].

We conlude this section with one more problem that should, in particular, have applications to methods of numerical integration. So far, the distributional properties (i.e., separation, covering and asymptotic behavior) are known for sets ω_N that are global minimizers for $E_s(\cdot)$ on $\mathbb{S}^d$; i.e., configurations $\omega_N \subset \mathbb{S}^d$ such that $E_s(\omega_N) = \mathscr{E}_s(\mathbb{S}^d;N)$. However, absolutely nothing is known about local minimizers.

Problem 6. *For every N fix a local minimizer $\omega_N \subset \mathbb{S}^d$ for $E_s(\cdot)$. Show that configurations ω_N are asymptotically optimal; i.e.,*

$$\lim_{N\to\infty} \frac{E_s(\omega_N)}{\mathscr{E}_s(\mathbb{S}^d;N)} = 1.$$

In particular, this will imply that

$$\frac{1}{N}\sum_{x\in\omega_N} \delta_x \xrightarrow{*} \frac{\mathscr{H}_d(\cdot\cap\mathbb{S}^d)}{\mathscr{H}_d(\mathbb{S}^d)}.$$

3. Discrete energy and polarization on compact sets for integrable Riesz kernels

Assume that $A\subset\mathbb{R}^p$ is a d-regular compact set with $0<\mathscr{H}_d(A)<\infty$. Let $K\colon A\times A\to\mathbb{R}$ be a lower semi-continuous symmetric function. Similarly to (11) and (12) we define

$$\mathscr{E}_K(A;N) := \min_{\omega_N} \sum_{x_i,x_j\in\omega_N, i\neq j} K(x_i,x_j)$$

and

$$\mathscr{P}_K(A;N) := \max_{\omega_N} \min_{y\in A} \sum_{x_i\in\omega_N} K(x_i,y).$$

It is known [9], [29] that there exist constants $W_K(A)$, the *K-Wiener constant*, and $T_K(A)$, the *K-Chebyshev constant*, possibly infinite, such that

$$\lim_{N\to\infty}\frac{\mathscr{E}_K(A;N)}{N^2} = W_K(A), \quad \lim_{N\to\infty}\frac{\mathscr{P}_K(A;N)}{N} = T_K(A).$$

Moreover, we have

$$W_K(A) = \min_{\mu}\int_A\int_A K(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y), \quad T_K(A) = \max_{\mu}\min_{y}\int_A K(x,y)\mathrm{d}\mu(x),$$

where both suprema are taken over probability measures μ supported on A.

The *K-capacity* of the set A is defined by $\mathrm{Cap}_K(A) := 1/W_K(A)$. In particular, if $K(x,y) = |x-y|^{-s}$ with $s>d$, then $\mathrm{Cap}_K(A)=0$; therefore, results of the previous section do not fall in the potential theory framework. In this section we discuss the case when $\mathrm{Cap}_K(A)>0$. Suppose μ_N is a normalized N-point probability measure that minimizes $\mathscr{E}_K(A;N)$. For a large class of kernels K it is known that $\mu_N \xrightarrow{*} \mu^*$ and

$$\int_A K(x,y)\mathrm{d}\mu^*(x)\mathrm{d}\mu^*(y) = W_K(A)$$

Thus, μ^* minimizes the *continuous energy* on A. For a long time, limiting behavior of discrete measures that attain $\mathscr{P}_K(A;N)$ was unknown, except for the case $K\in C(A\times A)$

which does not include Riesz potentials $|x-y|^{-s}$, $s>0$. It was proved by Simanek [37] that if there exists a probability measure μ_{eq} with $\mathrm{supp}(\mu_{eq})=A$ and

$$\int_A K(x,y)\mathrm{d}\mu_{eq}(x)=C,\ \ \forall y\in A,$$

then μ_{eq} attains both $W_K(A)$ and $T_K(A)$, and $\mu_N \xrightarrow{*} \mu$, where μ_N are normalized N-point probability measures that attain $\mathscr{P}_K(A;N)$. When $A=\mathbb{S}^d$ or $A=\mathbb{B}_d$, the d-dimensional unit ball, these conditions are satisfied for some Riesz potentials; however, when $A=\mathbb{B}_d$, they are not satisfied for $K(x,y)=|x-y|^{-s}$ with $0<s\leqslant d-2$. In our recent paper with E. Saff and O. Vlasiuk, the following is shown.

Theorem 3.1. *Let $K(x,y)=f(|x-y|)$, where f is a decreasing continuous function on $(0,\infty)$ bounded from below. Assume further that for some ε with $0<\varepsilon<d$ and a number $t_\varepsilon>0$ the function $t\mapsto t^{d-\varepsilon}f(t)$ is increasing on $[0,t_\varepsilon]$, where the value at 0 is defined by*

$$\lim_{t\to 0^+} t^{d-\varepsilon}f(t).$$

Assume $A\subset\mathbb{R}^p$ is a d-regular compact set, $d\leqslant p$. Let $\{\mu_N\}$ be a sequence of normalized discrete probability N-point measures that attain $\mathscr{P}_K(A;N)$, and assume for some subsequence $\mathscr{N}$ we have $\mu_N\xrightarrow{*}\mu^*$, $N\in\mathscr{N}$. Then μ^* attains $T_K(A)$; i.e.,

$$\inf_{y\in A}\int_A K(x,y)\mathrm{d}\mu^*(x)=T_K(A).$$

Notice that this result is valid for $A=\mathbb{B}_d$ and any Riesz kernel $K(x,y)=|x-y|^{-s}$, $s>0$, as well as for $K(x,y)=\log(2/|x-y|)$. The following minimum principle is a key ingredient of the proof.

Theorem 3.2. *Let A and K satisfy the conditions of Theorem 3.1. If for a measure μ supported on A and some constant C we have*

$$\int_A K(x,y)\mathrm{d}\mu(x)\geqslant C \ \ \text{for } \mathscr{H}_d\text{-a.e. } x\in A, \tag{20}$$

then the inequality (20) holds for every $x\in A$.

We in particular see that we do not require any regularity properties of the measure μ. However, Theorem 3.1 can hold in more generality if we manage to take into account some structure of optimal measures μ_N.

Problem 7. *Extend Theorem 3.1 to more general family of kernels K. In particular, prove it (or give a counterexample) in the general case of lower semi-continuous kernel K and a compact set A with $\mathrm{Cap}_K(A)>0$.*

4. Crystallization phenomena

Questions of crystallization are another versions of problem about arrangements of particles, that minimize certain energy. However, we now do not want to necessarily restrict the particles to a fixed compact set. These questions can be summarized as follows: what causes the atoms to arrange themselves in a periodic way? More specifically, why do crystals consist of atoms arranged with respect to a certain lattice? These problems have great significance as evidenced by the fact that 2014 was named the “International Year of Crystallography" by UNESCO. However, it seems like no general theory has been developed to resolve these problems in dimensions $p > 1$. This is not only because these problems are difficult, but they were somewhat overlooked by mathematicians: indeed, most papers in this topic were published in physics journals.

For a potential $K\colon \mathbb{R} \to \mathbb{R}$ with $K(x,y) = f(|x-y|)$ and $f(t) \to 0$ as $t \to \infty$, and an N-point multiset $\omega_N \subset \mathbb{R}^p$, we define the N-point energy of ω_N and K by

$$E_K(\omega_N) = \sum_{i \neq j} f(|x_i - x_j|), \ \ \omega_N = \{x_1, \dots, x_N\}, \tag{21}$$

and

$$\mathscr{E}_K(N) := \inf_{\omega_N} E_K(\omega_N). \tag{22}$$

We assume that particles on small distances repel each other, but on large distances attract each other. The most famous potential that describes this behavior is the Lennard-Jones potential

$$K_{LR}(x,y) = c_6/|x-y|^6 - c_{12}/|x-y|^{12}.$$

On the other hand, if

$$K(x,y) = \begin{cases} +\infty, & |x-y| < 1 \\ -1, & |x-y| = 1 \\ 0, & |x-y| > 1 \end{cases}$$

then minimization of energy $\mathscr{E}_K(N) :=$ is equivalent to sphere packing problem. For this potential, in $\mathbb{R}^2$ the minimization configurations in the limit form hexagonal lattice, which arose in the previous problem. The general question of interest is: how are the minimizing configurations ω_N arranged as N tends to infinity? The known theorems are only for the case $p = 1$; i.e., only the case of real line $\mathbb{R}$, see [15], [38].

Problem 8. *Obtain sufficient conditions on K that guarantee that the limit*

$$e_K := \lim_{N \to \infty} \frac{\mathscr{E}_K(N)}{N}$$

is a finite number. Further, set

$$\mu_N := \sum_j \delta_{x_j^N},$$

where $\omega_N = \{x_1^N, \dots, x_N^N\}$ minimize $\mathscr{E}_K(N)$. Obtain conditions sufficient for the sequence $\{\mu_N\}$ to converge in the weak* sense to a measure supported on a lattice.

We mention a particular case of this problem. Suppose we assume that $\mu_N \xrightarrow{*} \mu$, where μ is a locally finite measure supported on a lattice. Can we tell which lattice is it? Even this question is not understood in $\mathbb{R}^2$.

5. Weighted estimates for Calderón–Zygmund operators

A Calderón–Zygmund operator is an integral operator whose kernel satisfies certain growth and cancellation conditions. The most famous examples of such operators are the Hilberg Transform

$$Hf(x) := p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} \mathrm{d}y, \ \ f \in C_0^\infty(\mathbb{R})$$

and the Riesz transform in $\mathbb{R}^d$

$$Rf(x) := p.v. \int\limits_{\mathbb{R}^d} \frac{x-y}{|x-y|^{d+1}} f(y)\mathrm{d}y, \;\; f \in C_0^\infty(\mathbb{R}^d).$$

For a harmonic function u on $\mathbb{R}$, the Hilbert transform is used to find it's harmonic conjugate v, thus the operator H plays an important role in the theory of holomorphic functions in the upper-half plane. Another important Calderón–Zygmund operator is the Ahlfors–Beurling transform

$$Bf(z) := p.v. \int\limits_{\mathbb{C}} \frac{f(w)}{(z-w)^2} \mathrm{d}m_2(w),$$

where m_2 denotes the Lebesgue measure on the complex plane $\mathbb{C}$; the Ahlfors–Beurling transform plays a very important role in the theory of quasiconformal maps. For all these operators, it is known that they are bounded from L^p to L^p when $1 < p < \infty$. They fail to be bounded from L^1 to L^1, however, they are bounded from L^1 to $L^{1,\infty}$. For the Hilbert transfor H it means that there exists a constant $C > 0$, such that for any $f \in L^1$ and any $t > 0$ we have

$$|\{x \in \mathbb{R} \colon |Hf(x)| \geqslant t\}| \leqslant C \cdot \frac{\|f\|_1}{t},$$

where for a set A we denote it's 1-Lebesgue measure by $|A|$. We proceed with the following definition.

Definition 2. We say that a positive function $w \colon \mathbb{R} \to \mathbb{R}$ belongs to A_p with $1 < p < \infty$ if

$$[w]_p := \sup_I \left(\frac{1}{|I|} \int\limits_I w(x)\mathrm{d}x \right) \cdot \left(\frac{1}{|I|} \int\limits_I w(x)^{-1/(p-1)}\mathrm{d}x \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$. We say that w belongs to A_1 if

$$[w]_1 := \sup_I \left(\frac{1}{|I|} \int\limits_I w(x)\mathrm{d}x \right) \cdot \left(\inf_I w \right)^{-1} < \infty.$$

Constant $[w]_p$ is called the A_p-*characteristic* of w.

It is known [18] that if $w \in A_p$ with $1 < p < \infty$, then H maps $L^p(w\mathrm{d}x)$ to $L^p(w\mathrm{d}x)$. For operators B, R and H it was proved in [33], [31], [32], and [25] that their $L^2(wdx)$ norms depend linearly on the A_2-characteristic of w. Further, the same result was proved in [20], [19], [27] for any Calderón–Zygmund operator. In my joint work with F. Nazarov and A. Volberg, I extended this result to general metric spaces. Theorem below can be applied, for example, to a Cauchy transform defined on a rectifiable curve.

Theorem 5.1. *Assume (A, ρ, μ) is a metric measure space with a doubling Borel measure μ; i.e., there exists a constant C, such that for any ball $B(x,r) := \{y \in A \colon \rho(x,y) \leqslant r\}$ we have*

$$\mu(B(x,r)) \leqslant C\mu(B(x,r/2)).$$

Let

$$[w]_{2,\mu} := \sup_{x \in A, r>0} \left(\frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} w(x)\mathrm{d}\mu(x) \right) \cdot \left(\frac{1}{\mu(B(x,r))} \int\limits_{B(x,r)} w(x)^{-1}\mathrm{d}\mu(x) \right) < \infty.$$

Then for any Calderón–Zygmund operator T there exists a constant $c(T)$, such that for any function f

$$\left(\int_A |Tf(x)|^2 w(x) \mathrm{d}\mu(x)\right)^{1/2} \leqslant c(T)\cdot [w]_{2,\mu}\cdot \left(\int_A |f(x)|^2 w(x) \mathrm{d}\mu(x)\right)^{1/2}.$$

We turn our attention to the case $p = 1$. It was proved in [18] that H is bounded from $L^1(w\mathrm{d}x)$ to $L^{1,\infty}(w\mathrm{d}x)$ if and only if w belongs to A_1. Moreover, in [30] the following quantitative upper bound was obtained:

$$w(\{x\in\mathbb{R}\colon |Hf(x)|\geqslant t\}) \leqslant C\cdot [w]_1\cdot \log(e+[w]_1)\cdot \frac{\|f\|_{L^1(w\mathrm{d}x)}}{t}, \tag{23}$$

where for a set $A\subset\mathbb{R}$ we define $w(A) := \int_A w(x)dx$. Since then it was conjectured that, in fact, the log factor can be erased. In particular, it was proved for decreasing weights w. A weaker conjectured was disproved in [35] and [36]. Together with F. Nazarov, V. Vasyunin and A. Volberg, I obtained the following.

Theorem 5.2. *For every large $Q > 1$ there exists a function w with A_1-characteristic equal to Q, a function f and a number $t > 0$, such that*

$$w(\{x\in\mathbb{R}\colon |Hf(x)|\geqslant t\}) \geqslant C\cdot Q\cdot (\log(e+Q))^{1/4}\cdot \frac{\|f\|_{L^1(w\mathrm{d}x)}}{t}. \tag{24}$$

Thus, the log *factor is essential.*

One can see that powers of $\log(e+Q)$ in upper and lower bounds are different. It will be interesting to obtain the sharp bound.

Problem 9. *Obtain a sharp estimate of the type* (23), (24)*; moreover, construct the weight w and function f that attain this sharp estimate.*

6. Toeplitz operators on the Bergman Space

Let $\mathbb{H} := \{z\in\mathbb{C}\colon \mathrm{Im}(z) > 0\}$ be the upper-half plane and m_2 be the Lebesgue measure on $\mathbb{C}$. We associate the boundary $\partial\mathbb{H} = \{z\in\mathbb{C}\colon \mathrm{Im}(z) = 0\}$ with the set of real numbers $\mathbb{R}$. For an interval $I\subset\mathbb{R}$ denote it's Carleson box by $B_I := I\times[0,|I|]$. Besides Carleson boxes, it is convenient to consider their upper-halves $U_I := I\times[|I|/2,|I|]$. Let us give a motivation. Fix a dyadic lattice $\mathscr{D}$ on $\mathbb{R}$; i.e.,

$$\mathscr{D} := \{[2^k\ell, 2^k(\ell+1)]\colon k,\ell\in\mathbb{Z}\}.$$

Then we have

$$\bigcup_{I\in\mathscr{D}} U_I = \mathbb{H},$$

and the sets U_I have no common interior points. Therefore, for a function f on $\mathbb{H}$ we have

$$f(z) = \sum_{I\in\mathscr{D}} \mathbb{1}_{U_I}(z) f(z),\ \ m_2 - \text{a.e.}$$

We further introduce the set $L^2_a(\mathbb{H}, m_2)$; i.e., the set of all holomorphic functions f on $\mathbb{H}$, such that

$$\int_{\mathbb{H}} |f(z)|^2 dm_2(z) < \infty.$$

For a function $u\colon \mathbb{H} \to \mathbb{C}$ define

$$T_u f(z) := \int\limits_{\mathbb{H}} \frac{f(\zeta)u(\zeta)}{(z-\bar{\zeta})^2} dm_2(\zeta).$$

It is well known that if $u \in L^\infty$ then T_u is bounded from $L^2_a(\mathbb{H}, m_2)$ to itself. Moreover, define

$$k_\zeta(z) := \frac{1-|z|^2}{(1-\bar{\zeta}z)^2};$$

in [2] it is proved that if $u \in L^\infty$ and the function

$$S(\zeta) := \int\limits_{\mathbb{H}} T_u k_\zeta(z) \overline{k_\zeta(z)} dm_2(z)$$

satisfies $S(\zeta) \to 0$ as $\mathrm{Im}(\zeta) \to 0$, then T_u is a compact operator.

Problem 10. *Obtain sufficient conditions on u, weaker than $u \in L^\infty$, for T_u to be bounded from $L^2_a(\mathbb{H}, m_2)$ to itself.*

So far, it is known that condition that $S(\zeta)$ is a bounded function is not sufficient. I have obtained the following partial result.

Theorem 6.1. *Assume for every $I \in \mathbb{D}$ and every rectangle $R \subset U_I$, such that left-bottom corners of R and U_I coincide, we have*

$$\left| \int\limits_R u(z) dm_2(z) \right| \leqslant |I|^2.$$

Then the operator T_u is bounded.

Notice that in the above condition the absolute values are outside the integral; thus, we are able to derive boundedness of T_u from some oscillation properties of u. However, we still have too many conditions to verify. Ideally, I want to prove that T_u is bounded if for every $I \in \mathscr{D}$ we have

$$\left| \int\limits_{U_I} u(z) dm_2(z) \right| \leqslant |I|^2.$$

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