Extremal problems for lattices with bounded VC dimension.

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first part is a joint work with Alexandre Albano

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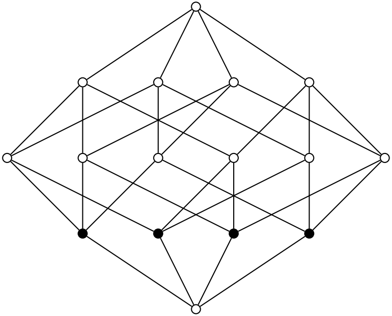
Motivation: exponential growth of lattices

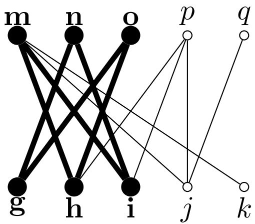


- We will be estimating the size of finite lattices with respect to the set of their join-irreducible elements.
- This size in general is obviously exponential: take B_n for example.
- The question is: are B_n's are the only reason for exponential blowup.

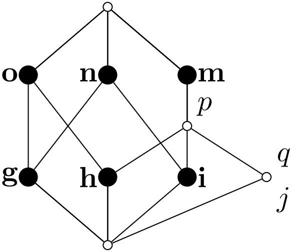
Prisner's bound

Theorem (Prisner, 2000, bicliques in bipartite graphs)

The bipartite graph $B = (U \cup W, E)$ without induced ***cocktail party graph CP(j)*** *contains at most $\left(|U||W|\right)^{j-1}$ maximal bicliques.*



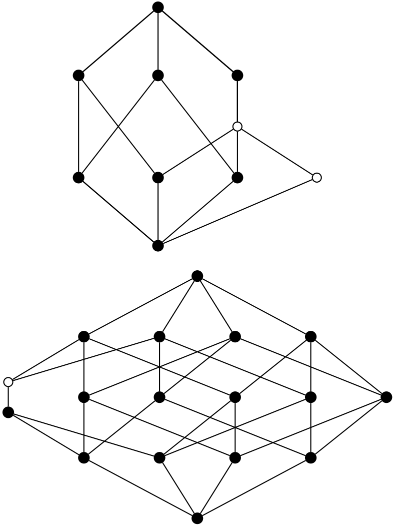
	m	**n**	**o**	p	q
g		×	×		
h	×		×	×	
i	×	×		×	
j	×			×	×
k	×				



As every bipartite graph corresponds to a formal context, this bound can clearly be restated for lattices.

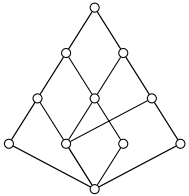
Alternative definition: order-embedding of B_k

The absence of large party graphs can be restated in different forms:



- As maximal order-embedded boolean lattice.
- This can be made join–embedding or meet-embedding, but not in general lattice-embedding.

Alternative definition: shattering and VC dimension



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- For the last definition, let us represent our lattice L as a family $\mathcal{F}$ of subsets of $J(L)$ - the set of join-irreducible elements of L.
- A subset $X \subseteq J$ is **shattered** by $\mathcal{F}$ iff $\mathcal{F}|_X = 2^X$.
- **Vapnik-Chervonekis** (VC) dimension of $\mathcal{F}$ is the size of maximal shattered set.
- This definition is given for arbitrary $\mathcal{F}$. The "canonical" way of representing lattices is by families of closed sets, that is, closed under the intersections. But the definition still applies.
- Luckily enough, all three definitions coincide.

Upper bound by Sauer-Shelah lemma

Lemma (Sauer-Shelah)

If $\mathcal{F}$ is a family of subsets of $[n]$ and $|\mathcal{F}| > f(n, k+1)$, then $\mathcal{F}$ shatters some $k+1$-set, where

$$f(n, k+1) = \sum_{i=0}^{k} \binom{n}{i}.$$

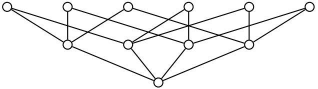
alternatively, for lattices

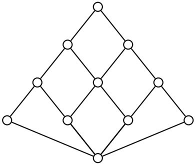
Lemma (Sauer-Shelah)

If VC-dimension of L is at most k, then $|L| \leq f(|J(L)|, k+1)$.

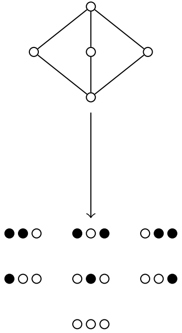
which is much better than the bound of Prisner: $|L| \leq (|J(L)||M(L)|)^k$.

Sauer-Shelah lemma, trivial and nontrivial directions



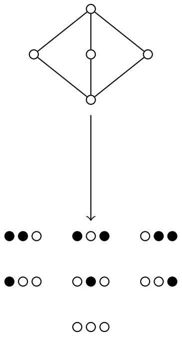


- For arbitrary set families, the "nontrivial" part is the upper bound given by Sauer-Shelah lemma.
- Its sharpness is obvious: it is reached on the family of all sets of size at most k.
- For lattices, however, the upper bound is easy to show, even without invoking the lemma (see next slide).
- But its sharpness is now not immediate, as the sharp family for the general case does not define a lattice.



- For an element $a \in L$, a set $S \in J(L)$ is a **minimal join-representation** of a if $a = \bigvee S$ and $a > \bigvee S'$ for any $S' \subsetneq S$.
- We denote the set of all minimal representation by $\mathrm{MJR}(L)$. An element can have many minimal join-representation, however a representation clearly uniquely determines an element, thus $|\mathrm{MJR}(L)| \geq |L|$.

Upper bound without Sauer-Shelah lemma



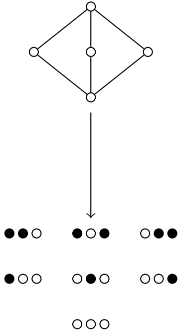
- The family $\mathrm{MJR}(L)$ is **hereditary**, and it is easy to show that

Lemma

For arbitrary finite lattice L,
$\mathrm{MJR}(L) = Sh(L)$*, where $Sh(L)$ is a family of subsets of $J(L)$, shattered by L.*

- Now, if $VC(L) \leq k$ then $f(|J|, k+1) \geq |Sh(L)| = |\mathrm{MJR}(L)| \geq |L|$. The first equation holds, because no $k+1$ set can be shattered.

Upper bound without Sauer-Shelah lemma



- In particular, we have a characterization of lattices, reaching our bound: those are the lattices where:
  - every element has a unique minimal join representation, and;
  - all k-sets of $J(L)$ are m.j.r-s.
- Lattices of VC-dimension at most k, with n join-irreducible elements, reaching the bound, we will call $(n, k+1)$-**extremal**. (We have not yet shown that those exist).

Background: finite meet-distributive lattices

A finite lattice L is called **lower semimodular** iff any covering is either preserved or collapsed by an arbitrary meet. L called **join semidistributive** iff $w = u \vee s = u \vee t$ implies $w = u \vee (s \wedge t)$.

L is **meet-distributive**, if one of the following equivalent conditions holds:

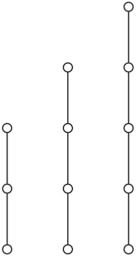
- L is join semidistributive and lower semimodular;
- L is a lattice of a **convex geometry**, that is, of closed sets of a closure space with **anti-exchange property** (to be discussed later);
- for all x, the interval $[x^*, x]$ is boolean, where x^* is a meet of lower neighbors of x;
- every element of L has a unique minimal join-representation;
- all maximal chains of L have length $|J(L)|$;
- and many more...

Later we will use yet another characterization in terms of implications.

Simple examples of extremal lattices

- An $(n, 1)$-extremal lattice is simply a one-element lattice, for all n;

Simple examples of extremal lattices



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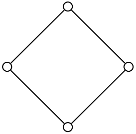

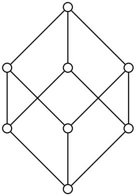

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- For $n \leq k$, $(n, k + 1)$-extremal lattice is just B_n;

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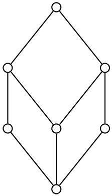

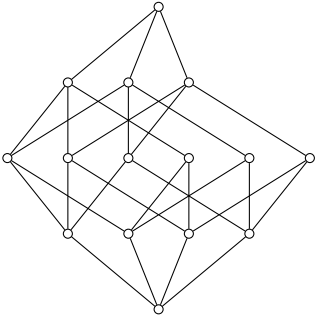

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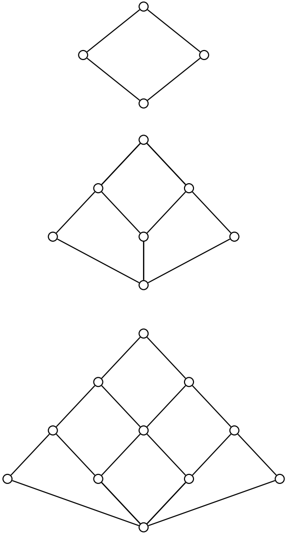
- (n, n)-extremal lattice is B_n with one coatom removed;

Simple examples of extremal lattices



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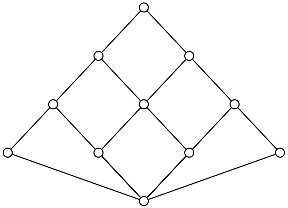
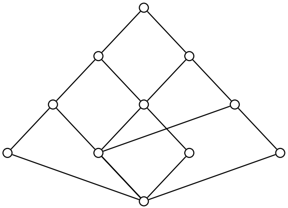
- A good example of $(n,3)$-extremal lattices are interval lattices;

Simple examples of extremal lattices





- An $(n, 1)$-extremal lattice is simply a one-element lattice, for all n;
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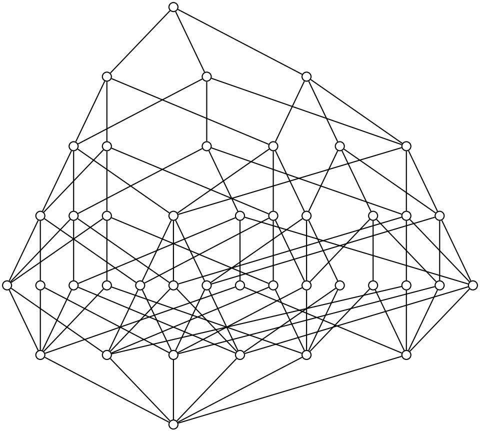
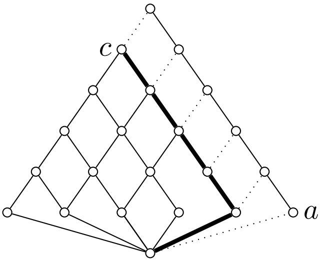
- (n, n)-extremal lattice is B_n with one coatom removed;
- A good example of $(n, 3)$-extremal lattices are interval lattices;
- But those are not the only examples;

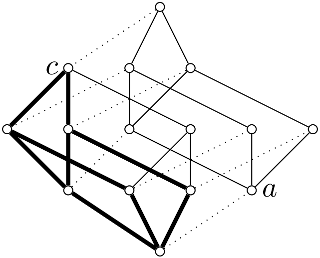
And not so simple

In general, construction with a picture may not be that accessible. We need some way to construct arbitrary extremal lattices.



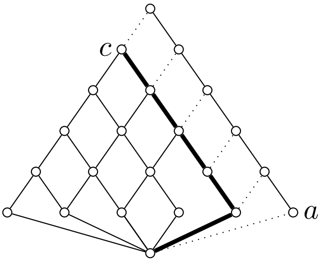
Deconstruction of extremal lattices

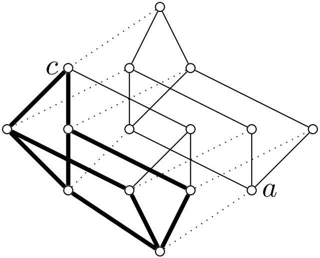




- Given an $(n+1, k+1)$-extremal lattice L, let us pick an arbitrary coatom c, then there is a unique atom a such that a is not below c;
- moreover, $L = (c] \sqcup [a)$;
- moreover, both $(c]$ and $[a)$ are meet-distributive;
- moreover, $(c]$ is $(n, k+1)$-extremal and $[a)$ is (n, k)-extremal;
- and the map $x \mapsto x \wedge c$ is a cover-preserving $(1, \wedge)$-embedding of $[a)$ into $(c]$;

Deconstruction of extremal lattices





In fact, the opposite also holds:

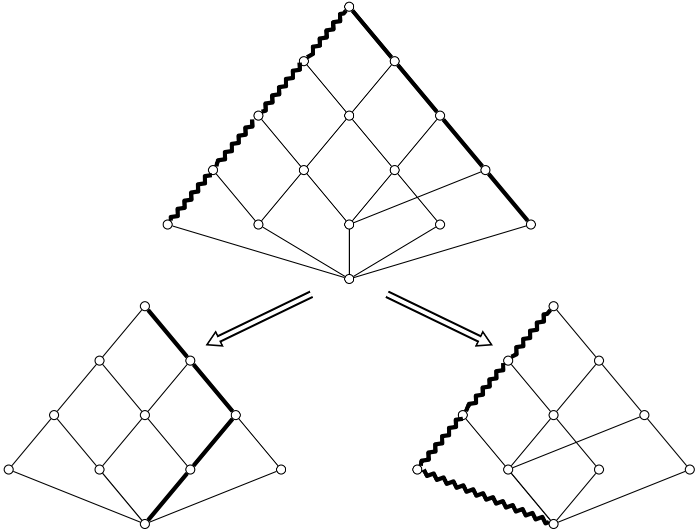
Lemma

Let L be a meet-distributive lattice and K be a cover-preserving, $(1, \wedge)$-subsemilattice of L. Then, **doubling** *$L[K]$ is a meet-distributive lattice.*

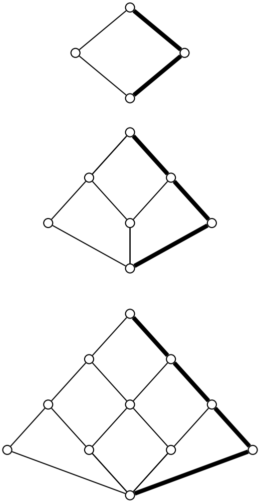
If L is $(n, k+1)$-extremal and K is (n, k)-extremal, then $L[K]$ is $(n+1, k+1)$-extremal.

So, we still do not know if arbitrary extremal lattices exist, but if they do, they can be constructed "from below" with doubling construction. Nice!

General note: deconstruction is not unique



Constructing $(n, 3)$-extremal lattices (easy)



- Pick any $(n, 3)$-extremal lattice L for n for which you know it exists. We can always start with $(2, 3)$-extremal lattice;
- pick arbitrary maximal chain C: by meet-distributivity of L, it will be of length n (i.e. with $n + 1$ element);
- as we know, C is $(n, 2)$-extremal;
- double C in L. Done! Now $L[K]$ is your $(n + 1, 3)$-extremal lattice. Now iterate the construction.
- Well, we knew already that interval lattices are extremal, but that is how we can construct arbitrary $(n, 3)$-extremal lattice.

Constructing arbitrary extremal lattices: final preparation

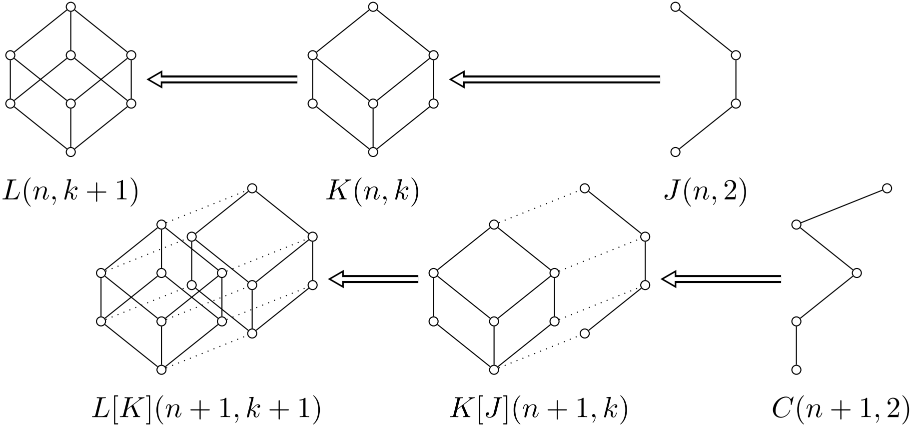
- So, in order for our iterative construction to work for all k, we need the ability to find $(n, k-1)$-extremal lattice K in a given (n, k)-extremal lattice L. How do we do it?
- Well, let us construct L such that it would already have K embedded into it.

Lemma

Suppose that J, K and L are lattices such that K is $(1, \wedge)$-embedded into L, and J is $(1, \wedge)$-embedded into L. Then, there exists a $(1, \wedge)$-embedding of $K[J]$ into $L[K]$.

The existence of arbitrary extremal lattices is now an easy corollary: let us look at the picture.

Constructing arbitrary extremal lattices
Proof by picture



Several minor questions

Question

We know that doubling construction is universal, however in our proof we used more specific construction, namely, doubling a chain of embedded lattices, and picking a new maximal chain on last iteration. Is this construction universal? (Probably no).

Question

Picking arbitrary $(n, 2)$*-extremal lattice inside a given* $(n, 3)$*-extremal lattice was easy: it was just an arbitrary maximal chain. But maybe there is an easy direct construction which would allow us to pick arbitrary (or enumerate all)* (n, k)*-extremal lattice inside a given* $(n, k+1)$*-extremal? (Probably yes).*

Generalization

Different extremalities, same energy

Sauer-Shelah, generalized

Let us look again at Sauer-Shelah lemma:

Lemma (Sauer-Shelah)

A family $\mathcal{F}$ of VC-dimension at most k has at most $\sum_{i=0}^{k} \binom{n}{i}$ elements.

or, restated

Lemma (Sauer-Shelah)

A family $\mathcal{F}$ which does not shatter any $k+1$-set has at most as many elements as family of all sets of size less than $k+1$.

Can we substitute "all $k+1$ sets" by arbitrary antichain? Yes, we can.

Sauer-Shelah, generalized

Lemma (Sauer-Shelah, generalized)

A family $\mathcal{F}$ which does not shatter an antichain $\mathcal{A}$ has at most as many elements, as $\mathcal{I}(\mathcal{A})$.
Here $\mathcal{I}(\mathcal{A})$ is a hereditary family of sets, which do not contain any set from $\mathcal{A}$, that is

$$\mathcal{I}(\mathcal{A}) = \{X \mid X \not\supseteq A, \text{ for all } A \text{ in } \mathcal{A}\}.$$

another way of formulating this is

Lemma

Every family $\mathcal{F}$ shatters at least as many elements as it has.

Families for which these inequalities become equalities (any of them will do, some clarification is coming) are called **shattering-extremal**.

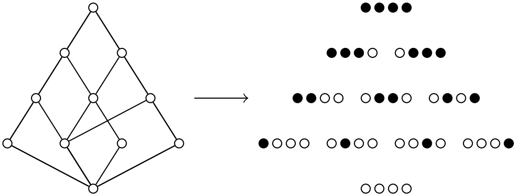
Note on terminology

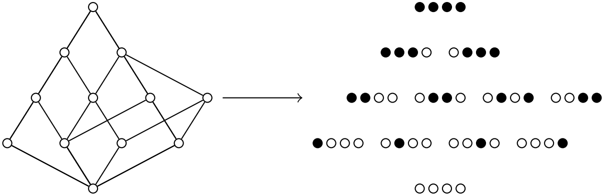
- Shattering-extremal families in their general form were first studied by Jim Lawrence(*Lopsided sets and orthant-intersection by convex sets*, 1983) and Béla Bollobás and A.J. Radcliffe (*Defect Sauer results*, 1992);
- Lawrence called them *lopsided sets*, his definition was not at all resembling our one. The equivalence was shown by Shay Moran in his master thesis (*Shattering extremal systems*, 2012);
- Bollobás and Radcliffe called them *extremal for Sauer*, their definition is essentially same as ours;
- Term *shattering-extremal* appears independently in Moran(2012) and in the dissertation of Tamás Mészáros(*Algebraic phenomena in combinatorics: shattering-extremal families and the combinatorial nullstellensatz*, 2015).

Back to antichains

- First, let us note that for a given family $\mathcal{F}$, there is an obvious way to associate a **blocking antichain** $\mathcal{A}_{\mathcal{F}}$ with it, such that if $\mathcal{F}$ is extremal for some chain, then it is extremal for $\mathcal{A}_{\mathcal{F}}$. Namely, take $\mathcal{A}$ to be an antichain of all minimal non-shattered sets;
- Now, if an antichain $\mathcal{A}$ is given, we can ask the opposite: is it extremal for some object;
- This question is clearly meaningless for arbitrary set family, as $\mathcal{A}$ is always extremal for $\mathcal{I}(\mathcal{A})$;
- For lattices, though the answer is not immediate, so
- An antichain $\mathcal{A}$ is **extremal** if it is extremal for some lattice (or closure family) L.

Just some pictures





- Both lattices are extremal, first one in the first sense, second one in generalized sense;
- The first blocking antichain is $\mathcal{A}_3$ - an antichain of all 3-sets. Any $\mathcal{A}_k$ is extremal;
- The second blocking antichain is $\{123, 124, 134\}$.

Characterization of extremal lattices (no surprises)

Theorem

- *Lattice L is extremal iff it is meet-distributive;*
- *closure system $\mathcal{C}$ is extremal iff it is a convex geometry.*

Knowing what we know, this is almost straightforward. Indeed:

- extremality means that lattice shatters as many sets as it has elements;
- which means that minimal join representations are unique;
- which means that lattice is meet-distributive;
- which means that it is a lattice of closed sets of a convex geometry.

There are some subtleties because not every closure system, giving rise to a meet-distributive lattice, is a convex geometry, but those can be dealt with.

Paying debt: convex geometries

- Convex geometries are families of closed sets, satisfying anti-exchange property (AEP): for all $x \neq y$ and all closed sets A, holds
$$x \in \overline{A+y} \text{ and } x \notin A \text{ imply } y \notin \overline{A+x}.$$
- The name "anti-exchange property" mirrors the *exchange property* (EP), which characterizes matroids: for all $x \neq y$ and all closed sets A, holds
$$x \in \overline{A+y} \text{ and } x \notin A \text{ imply } y \in \overline{A+x}.$$
- While matroids generalize linear dependence, convex geometries generalize families of convex sets.
- Examples: convex hulls on $\mathbb{R}^N$ or on arbitrary subset of $\mathbb{R}^N$, convex hulls on posets, families of paths on a tree.

Paying debt: convex geometries

- Convex geometries are often studied together with *antimatroids*, which can be defined as families of complements of convex geometries;
- The terminology is further mirrored by the fact that matroids at times go by the name of *combinatorial geometries*. Matroids and antimatroids are covered by a generalized construction called *greedoid* (Bernhard Korte, László Lovász. *Greedoids*, 1982);
- Convex geometries were first introduced by Robert P. Dilwoth (*Lattices with unique irreducible decomposition*, 1940). Next big study was conducted by Paul H. Edelman and Robert E. Jamison (*The theory of convex geometries*, 1985).
- The "state of the art" review is a chapter *Convex Geometries* of Kira Adaricheva and J.B. Nation in the second volume of *Lattice theory: special topics and applications* (George Grätzer, Friedrich Wehrung (eds.), 2016).

Characterizing extremal antichains

How hard it is to characterize extremal antichains?

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Well, it is hard.

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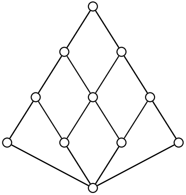
NP-hard!

Almost shattering



●○● $13 \to 2$
●○ ● $14 \to 2$
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●○● $24 \to 3$

- We say that $\mathcal{F}$ **almost shatters** a set X if $\mathcal{F}|_X = 2^X - 1$, that is, there is a unique subset of X not in the projection of $\mathcal{F}$. We will call such X's **forbidden projections**.
- If $\mathcal{F}$ is extremal, then it almost shatters every set of the blocking antichain $\mathcal{A}_{\mathcal{F}}$;
- For arbitrary $\mathcal{F}$ this does not in general hold. But there are examples of non-extremal families, almost shattering the blocking chain, that is, the converse of the previous statement does not hold in general.
- Additionally, if $\mathcal{F}$ is a closure system, for every forbidden projection X_A on A holds: $X_A = A - x$, for some $x \in X$. These forbidden projections, thus, can be understood as implications, holding in the lattice.

Forbidden projections of extremal lattices.

Checking if whether $\mathcal{A}$ is extremal can thus go as follows:

- for every $A \in \mathcal{A}$ pick an element $x_A \in A$, thus defining a forbidden projection $X_a = A - x_A$ (we will write it as $(A, x_A) \in \mathcal{A}$);
- construct a lattice generated by implications $X_a \to x_a$;
- check if this lattice is meet-distributive.

Apart from the necessity to check all possible choices of x_A's, the second step of this procedure is rather tiresome. Can we eliminate it?

Lemma (Brenda L. Dietrich, 1987)

An antichain $\mathcal{A}$ with chosen x_A's defines a convex geometry iff the following condition holds:

$$\text{if } a \in B - b \text{ then there is } (C, b) \in \mathcal{A} \text{ such that } C \subseteq A \cup B - a,$$

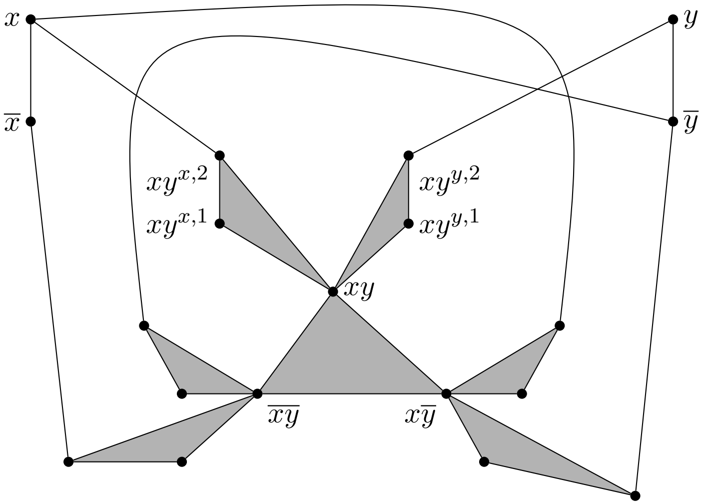
for all $(A, a), (B, b) \in \mathcal{A}$.

In fact, the original statement was formulated in terms of rooted circuits for antimatroids, but it is essentially the same.

NP-hardness (and NP-completeness) of extremal antichains

- With this characterization, checking if the given choice of x's produces a convex geometry for $\mathcal{A}$ is polynomial;
- Thus, checking if $\mathcal{A}$ is extremal is in NP (as we have to nondeterministically guess $x's$ first). Thus, proving that it is NP-hard would mean that it is actually NP-complete;
- To prove NP-completeness, we polynomially reduce 3SAT to it. In fact, every 3SAT formula is reduced to special kind of an antichain, which we call **intransitive**;
- We call $\mathcal{A}$ *intransitive* if for any distinct $A, B \in \mathcal{A}$ and for any point $x \in A \cap B$ there is no $C \in \mathcal{A}$ such that $C \subseteq A \cup B - x$;
- Such chains are convenient for us because every assignment of forbidden projections which produces a convex geometry, given any point x will either allow x in all projections, or forbid it in all projections.

NP-hardness. Proof by picture



Antichain fragment, corresponding to a formula $x \vee \neg y$.

Questions about extremal antichains

NP-completeness of extremal antichain means that we hardly will come up with some nice general characterization for them. But still:

Question

Do extremal antichains form a lattice under refinement? Does it have some "nice" properties? Do all its maximal chains have same lengths?

We know that all $\mathcal{A}_k$'s are extremal. Can we say something sensible about other "dense" antichains? For example:

Question

Given $l < k$, is there an antichain $\mathcal{A}_{l,k}$ of l-sets, such that every k-set contains some subset of $\mathcal{A}_{l,k}$ (that is, $\mathcal{A}_{l,k}$ is a Túran (n, k, l) system), such that $\mathcal{A}_{l,k}$ is "asymptotically minimal" and extremal.

This last problem seems to be ridiculously complex, as it is a wide open problem for what it means to be "asymptotically minimal" (reward of Erdös for any nontrivial k and l).

- A family $\mathcal{F}$ **strongly shatters** a set S if there is $P \subseteq X - S$ such that for a family $\mathcal{F}_P = \{F \in \mathcal{F} \mid F|_{X-S} = P\}$, holds $\mathcal{F}_P|_S = 2^S$;

Theorem (Bollobás, Radcliffe, 1992)

For a family $\mathcal{F}$ it holds $|SSh(\mathcal{F})| \leq |\mathcal{F}| \leq |Sh(\mathcal{F})|$, and if one of these inequalities is an equality, then so is the other.

- For convex geometries we get it for free, as strong shattering is equivalent to "the interval $[x^*, x]$ is a boolean lattice", which is one of our alternative definitions of meet-distributivity;
- This "for free" is subjective, as some subtleties have to be addressed, we have to have all these alternative definitions, plus the original theorem itself is not that hard;
- What is more interesting is whether we can pull some characterization from convex geometries to arbitrary shattering-extremal systems.

Let us recall forbidden-projections characterization for convex geometries:

Lemma (Brenda L. Dietrich, 1987)

An antichain $\mathcal{A}$ with chosen x_A's defines a convex geometry iff the following condition holds:

$$\textit{if } a \in B - b \textit{ then there is } (C, b) \in \mathcal{A} \textit{ such that } C \subseteq A \cup B - a,$$

for all $(A, a), (B, b) \in \mathcal{A}$.

Can we prove something similar for shattering-extremal systems?

Yes, we can:

Lemma

Let $\mathcal{S} = \{(S_i, h_i)\}$ be a family of forbidden projections. Then it is shattering-extremal iff the following condition holds:

for any $x \in \mathbf{Dif}(A, B)$ there is $(C, h_C) \in \mathcal{S}$ such that $C \subseteq \mathbf{Sup}(A, B) - x$, and h_C agrees with A and B on $\mathbf{Agr}(A, B) \cap C$,

for all $(A, h_A), (B, h_B) \in \mathcal{S}$.

Here h_i's are characteristic functions of forbidden projections, $h_i \colon S_i \to \{0, 1\}$, $\mathbf{Sup}(S, T) = S \cup T$ is a support of S and T, $\mathbf{Dif}(S, T) \subseteq S \cap T$ is a disagreement set of of h_S and h_T, and $\mathbf{Agr}(S, T)$ is an agreement set of h_S and h_T, $\mathbf{Agr}(S, T) = \mathbf{Sup}(S, T) - \mathbf{Dif}(S, T)$.

- Tamás Mészáros in his PhD work poses this nice problem:

Problem

Is every shattering-extremal system $\mathcal{F}$ ***eliminable****, that is, is there an $S \in \mathcal{F}$ such that $\mathcal{F} - S$ is shattering-extremal?*

- Actually, in private conversation he told that there seems to be a counterexample, for $\mathcal{A}_4$ with $n = 12$, originating from the dissertation of Huntington Tracy Hall (*Counterexamples in Discrete Geometry*, 2004). BTW, Vaughan Jones was in the committee.
- Still, it is easy to show that it holds for convex geometries.

Lemma

For a meet-distributive lattice L, the elements which can be removed without breaking the property of being meet-distributive lattice are exactly meet-irreducible elements, not covering any other meet-irreducible element.

The proof is very straightforward, but it relies on the notion of meet-irreducible element, for which there is no obvious counterpart in arbitrary shattering-extremal system.

Question

For which characterizations of convex geometries there are analogues in shattering-extremal systems?

To be continued...

The land of unknown

Estimating the number of meet-irreducible elements

- In our analysis we relied on the size of the set $J(L)$ of join-irreducible elements of a lattice;
- However, it is worth estimating the size of L not only w.r.t $|J(L)|$, but also w.r.t. to the set $M(L)$ of its meet-irreducible elements (or their combination). For example, in FCA the natural size of the description of L is the size of its canonical formal context, which is $|J| \cdot |M|$;
- so the natural question can sound like:

Question

What is the maximal size of a lattice L with $VC(L) \leq k$ and such that $\varphi(|J(L)|, |M(L)|) \leq n$. Which are the extremal objects?
Here $\varphi(n, m)$ can be something like $\alpha n + \beta m$ or $n^{\alpha} \cdot m^{\beta}$ with $\alpha + \beta = 1/2$.

Question

What is the maximal size of a lattice L with $VC(L) \leq k$ and such that $\varphi(|J(L)|, |M(L)|) \leq n$. Which are the extremal objects?

or simplier

Question

What is the maximal size of a lattice L with $VC(L) \leq k$ and such that $|J(L)| \leq n$ and $|M(L)| \leq n$. Which are the extremal objects?

even more simplier

Question

What is the minimal number of meet-irreducible elements of an $(n, k+1)$-extremal lattice L?

We will be able to answer a "fragment" of the last question, and only asymptotically, only for $k = 3$ and only under additional assumption.

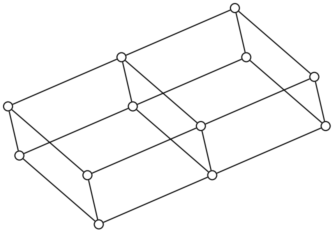
Symmetric case hypothesis

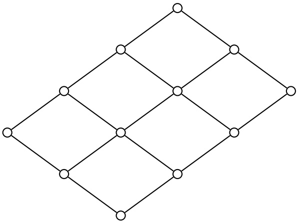
Conjecture

A lattice with VC-dimension at most k with $|G| + |M| = 2n$ has at most $\left(\frac{n}{k}\right)^k$ elements, and the extremal object is a distributive lattice

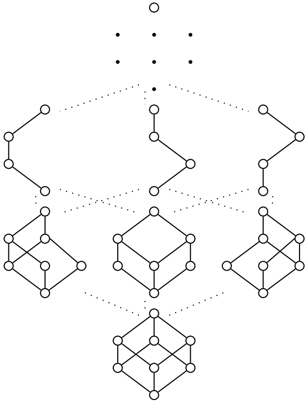
$$\bigtimes_k C\left(\frac{n}{k}\right),$$

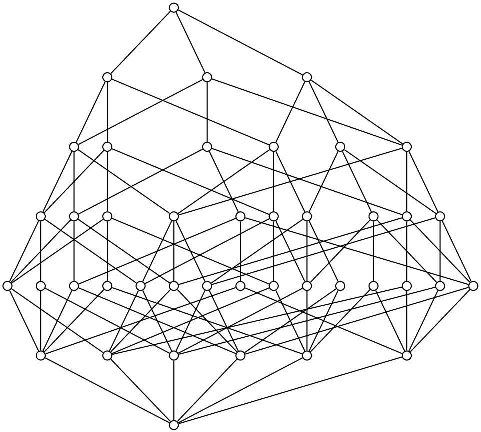
where $C(l)$ is an l-element chain.



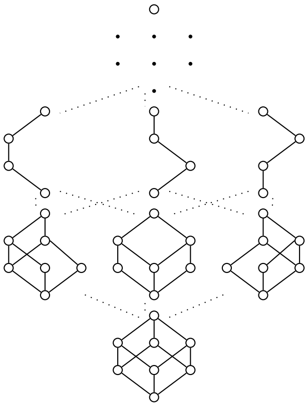


Decomposition of extremal lattices
Theorem by picture





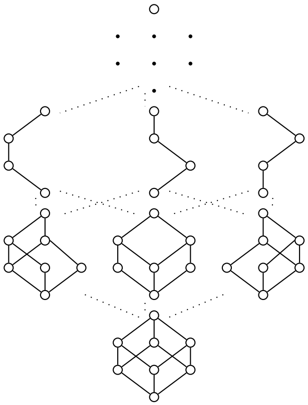
Decomposition of extremal lattices
Theorem by picture



Vaguely, theorem is formulated as follows

- An $(n+k, k+1)$-extremal lattice L can be decomposed into a B_k-shaped commutative diagram of lattices;
- All arrows are downward and are $(1, \wedge)$-inclusions;
- All lattices in nodes correspond to disjoint intervals of L
- ... and are $(n, k-h+1)$-extremal, where h is a height of the element in the diagram;
- This decomposition is unique up to permutations of the diagram B_k.

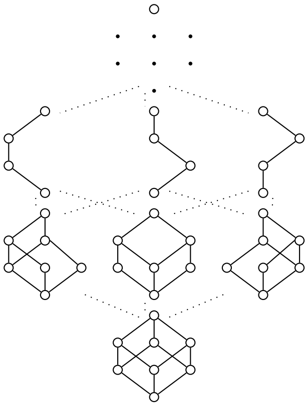
Decomposition of extremal lattices
Theorem by picture



This theorem is technical, but very useful:

- It is now much easier to visualize "moderately big" extremal lattices;
- Due to uniqueness, it provides a loot of "control" over how extremal lattices are constructed. For example, with it we can come up with a simple recursive formula for the number of different $(n, 3)$-extremal lattices, up to isomorphism;

Decomposition of extremal lattices
Theorem by picture



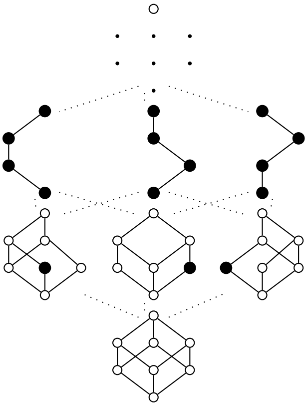
- Finally, it enables us to access meet-irreducible elements:

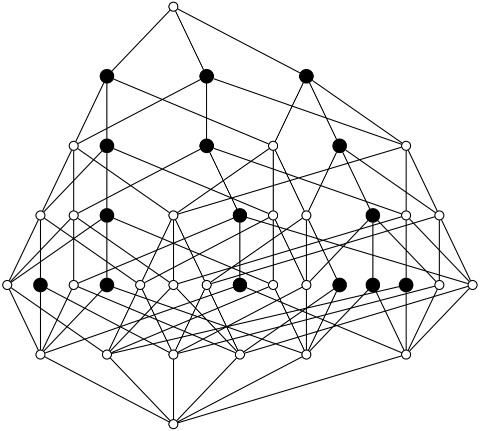
Lemma

An element (X, x) of a decomposition $\L$ is meet-irreducible iff

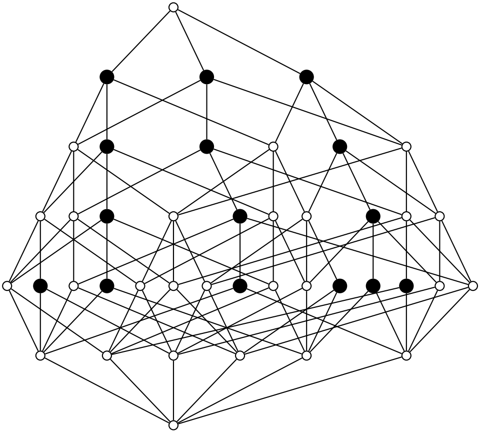
- *either x ix meet-irreducible in L_X and not presented in L_Y for any $Y \supsetneq X$;*
- *or x is a unit of a coatom in the decomposition.*

Decomposition and m.i. elements





Lower bound on the number of m.i. elements

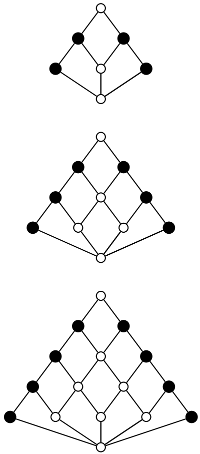


Lemma

Any $(n+k, k+1)$-extremal lattice L has at least $k(n+1)$ meet-irreducible elements, arranged in k disjoint chains of length n each. Every such chain contains exactly one element of rank i, for $i \in k-1, \ldots n+k-1$.

We call these **principal chains**, and their elements **principal meet-irreducible elements**.

Lower bound on the number of m.i. elements



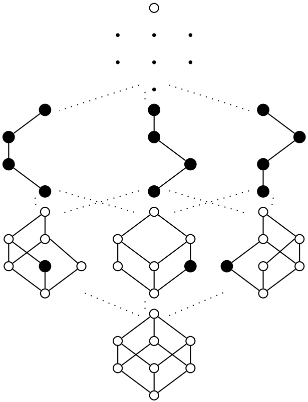
- We call extremal lattices having a minimal number of meet-irreducible elements **doubly extremal**;
- As interval lattices have no other meet-irreducible elements except for principal chains, we get:

Corollary

Interval lattices are $(n, 3)$-doubly extremal.

- all other $(n, 3)$-extremal lattices have some other meet-irreducible elements, which means that interval lattices are the only $(n, 3)$-doubly extremal.

Non-principal meet-irreducible elements



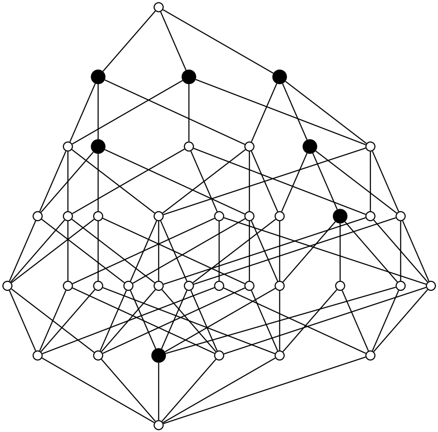
For larger k, however, there will be other meet-irreducible elements:

Conjecture

- *There is a way of constructing $(2*k-1, k+1)$-extremal lattice with only principal meet-irreducible elements;*
- *but for $k \geq 3$, an $(2*k, k+1)$-extremal lattice will have at least k meet-irreducible elements apart from principal chains.*

K-orderings problem

- Instead of trying to take non-principal meet irreducible elements into account, we will concentrate on principal chains, and try to decide, which fraction of a lattice they can generate by intersections.
- Notice that the fragment generated by principal chain will still be meet-distributive, and thus extremal in generalized sense.
- Every principal chain corresponds (almost) to a linear ordering of the elements in the base set.

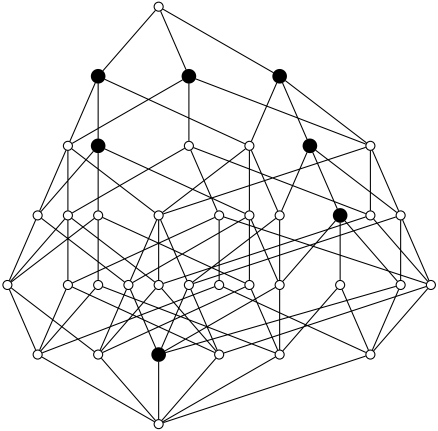


K-orderings problem

Problem (k-orderings problem)

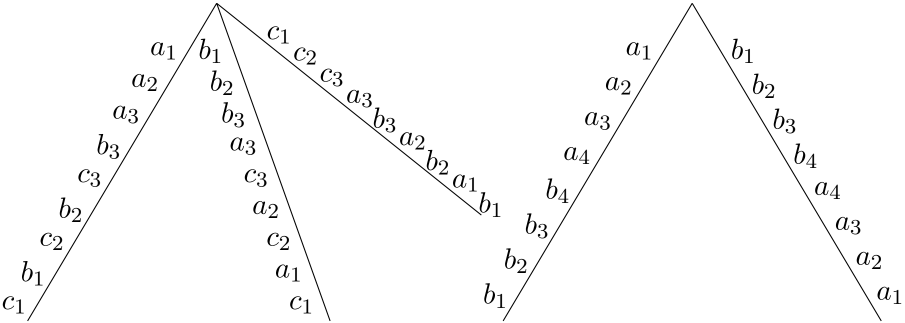
*How to construct k linear orderings of n elements, to maximize the number of **feasible** sets, that is, of sets, obtained as intersections of the initial intervals of these orderings.*

We only care about the asymptotics, for a fixed k, as n goes to infinity.



Tentative solution

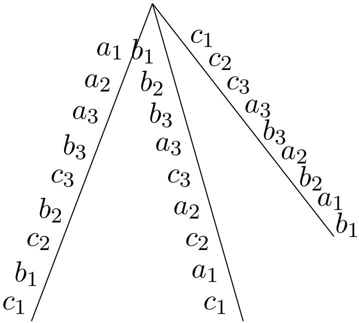
We start from the tentative solution.



Here a_i, b_i and c_i is simply an “arbitrary” subdivision of the base set $\mathbf{n}$.

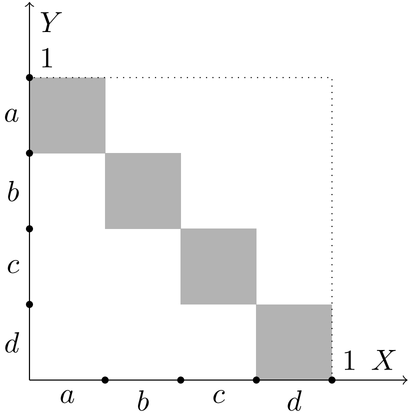
Feasible sets

- A k-tuple (a, b, c) is **feasible** if $a \leq_A b, c$, $b \leq_B a, c$ and $c \leq a, b$, where $\leq_A$, $\leq_B$ and $\leq_C$ are corresponding orderings of $\mathbf{n}$.
- Similarly, a k-set is **feasible** if it corresponds to some feasible tuple. There can be at most one such tuple, but can be none.
- We, thus, aim at maximizing the number of feasible sets, or, alternatively, the number of feasible tuples.
- On the picture, the set $\{a_2, c_1, c_3\}$ is feasible, corresponding to the tuple (a_2, c_3, c_1). The set $\{a_1, a_2, a_3\}$ is unfeasible.



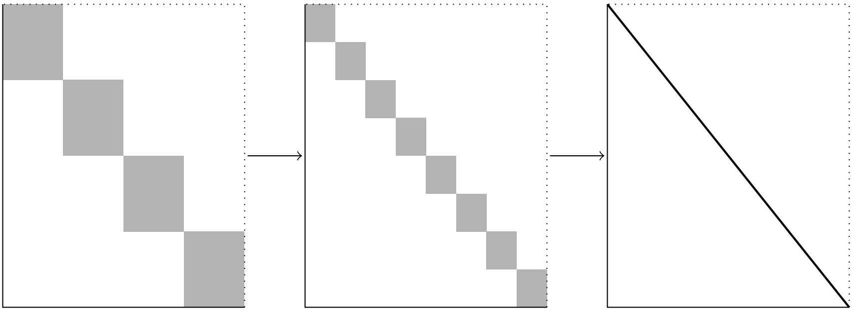
A way to draw 2 orderings

- Let us take two orderings on **4**: $a \leq b \leq c \leq d$ and $d \leq c \leq b \leq a$;
- Let the interval $[0, 1]$ on x-axis correspond to the first ordering and on y to the second;
- Elements a, b, c and d correspond to intervals of length $1/4$ on the axes, and go in the corresponding order from the origin;
- Now on $[0, 1]^2$ let us "paint out" squares, corresponding to elements a, b, c and d.



A way to draw 2 orderings

Now let us try to increase the number of elements:

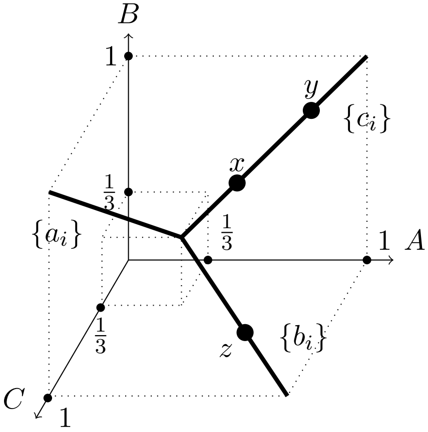


Intuitively, we want the last picture to represent the asymptotics of our extremal object. Our next step is motivated by construction of *graphons* (*Large Networks and Graph Limits*, László Lovász, 2012).

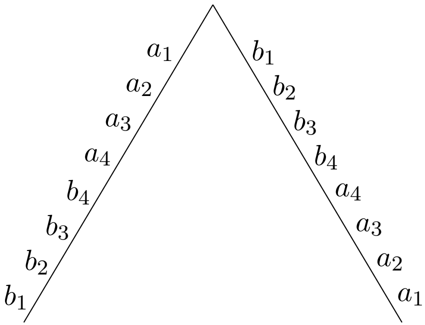
Limit configuration

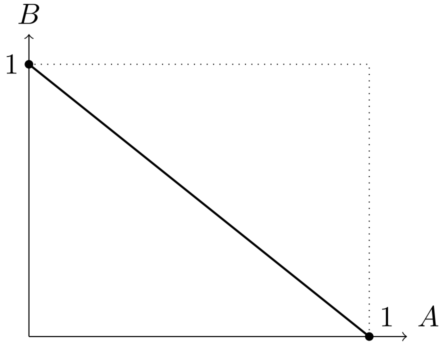
As long as we are interested in asymptotic behavior, it is convenient to introduce special objects, which capture the asymptotics.

1. **Limit configuration** is a measure μ on $[0,1]^k$, such that for every measurable set $B \subseteq [0,1]$, every $j = 1, \dots k$, it holds: $|B| = \mu\left(\pi_j^{-1}[B]\right)$.
2. A k-tuple $(x_1, \dots, x_k)$, $x_i \in [0,1]^k$, is **feasible**, if $\pi_i(x_i) \le \pi_i(x_j)$, for all i, j. $\mathcal{F} \subseteq [0,1]^{k^2}$ is the set of all feasible tuples.
3. The **volume** $vol(\mu)$, which we maximize, is thus defined as $vol(\mu) = k! \cdot \mu^k(\mathcal{F})$.

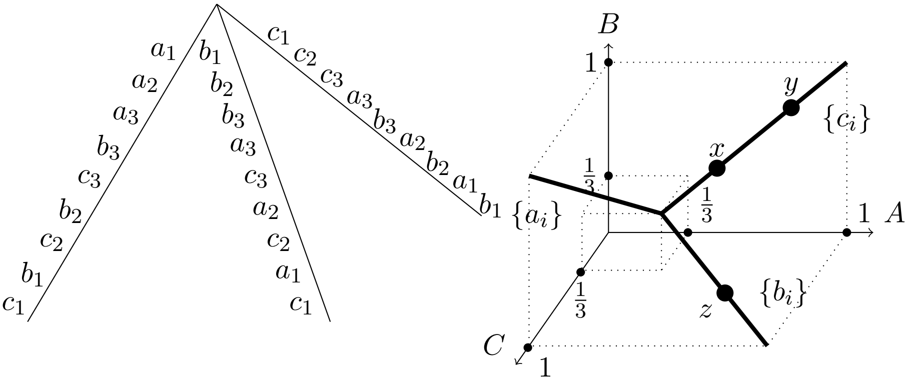


2-configuration





The volume of optimal 2-configuration is 1, which is maximum possible by a naive estimation. The corresponding lattice family are the interval lattices, which have no other meet-irreducible elements, except for the principal chains.



Optimal configuration and limit configuration $\mathcal{O}_3$ for $k = 3$. The triple $\{x, y, z\}$ is a feasible set, as long as $\pi_A(x) \leq \pi_A(z)$. $vol(\mathcal{O}_3) = \frac{2}{3}$. For a general k, $vol(\mathcal{O}_k) = k!/k^{k-1}$.

Why is it plausible: elementwise optimality

- In order for configuration to be optimal, all elements in $\mathbf{n}$ should "equally" contribute to the resulting volume: otherwise, if a contributes noticeably more than b, we can replace b with a copy of a, increasing the overall volume.
- In terms of limit configurations, the contribution $\omega(x)$ of an element $x \in [0,1]$ can be calculated as:

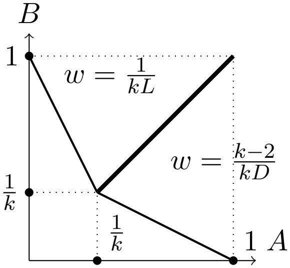
$$\omega(x) = (k-1)! \cdot \mu^{k-1}\big(\{(x_1, \ldots, x_k) \in \mathcal{F} \mid x_j = x\}\big).$$

- The necessary condition for a configuration to be optimal is thus that $\omega = C$ almost everywhere on $[0,1]$, for some fixed C. This condition holds for the optimal k-configuration $\mathcal{O}_k$ with $\omega(x) = k!/k^{k-1}$.

Why is it plausible: Kendall-Tau optimality

- We can also expect that the optimal chains will be "spread apart" from each other, as far as possible. In order to check this, we estimate the upper bound on the minimal pairwise **Kendall-Tau distance**, and see how our solution conforms to it.
- In terms of limit configurations, Kendall-Tau distance between chains A and B is simply the volume of the configuration $\mathcal{C}_{i,j}$, where $\mathcal{C}_{i,j}$ is a **projection** of $\mathcal{C}$ to the corresponding coordinates.

- It can be shown, that the optimal Kendall-Tau distance is $D_k = 2\frac{\lfloor k/2 \rfloor \lceil k/2 \rceil}{k(k-1)} \to 1/2$.
- For our solutions, however, KT-distance between the chains is $dist_{kt}^{i,j}(\mathcal{O}_{k,\infty}) = \frac{2}{k}$. It is optimal for $k = 2$ and $k = 3$, but not optimal for larger k.

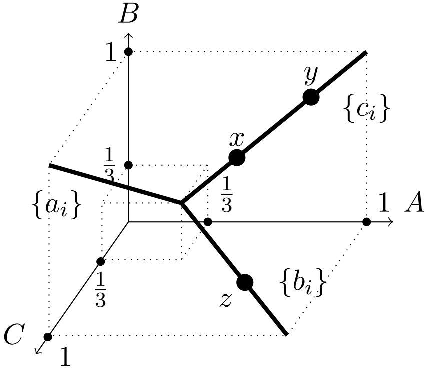


Solution for the symmetric case

Yet again, we find it natural for an optimal solution to be symmetric, as our tentative solution clearly is. Assuming this additional property, we are able to prove that our solution is optimal for $k = 3$.

Definition

A limit k-configuration (or simply a measure) μ is **symmetric** if $\mu(\rho_\sigma[X]) = \mu(X)$, for every permutation σ on $\mathbf{k}$ and every measurable $X \subseteq [0,1]^k$, where $\rho_\sigma\colon [0,1]^k \to [0,1]^k$ is a coordinate permutation function: $\rho_\sigma(x_1, \ldots, x_k) = (x_{\sigma(1)}, \ldots, x_{\sigma(k)})$.



The proof of the optimality is split into two main steps

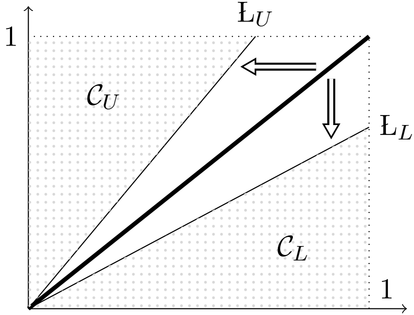
Solution for the symmetric case: freeing the diagonal

Our first step is preliminary: later it will be more convenient for us to deal with measure which is concentrated only in the set of points in "general position". Our tentative solution obviously does not possess such property.

Luckily, it can be done by pulling the measure a bit.

Lemma (Dediagonalization)

For a symmetric k-configuration μ there is a family $\{\mu_a\}_{a\in(1,\infty)}$ of symmetric continuous on projections diagonal-free measures on $[0,1]^k$ of total size 1, such that $\lim_{a\to 1} vol(\mu_a) = vol(\mu)$.



Solution: representing points with orbits

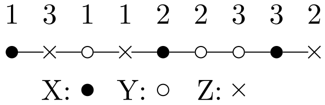
- Let us pick 3 points in $[0,1]^3$ in "general position" - dediagonalization lemma enables us to consider only such points:

$$\begin{cases} x = (0.1, 0.5, 0.8) \\ y = (0.3, 0.6, 0.7) \\ z = (0.4, 0.9, 0.2) \end{cases}$$

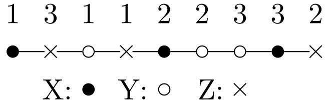
- To such 3-tuple we associate an **orbit**, that is a 3^3-tuple $o(x, y, z)$ of their coordinates represented as formal letters, but ordered by their original order:

$$o(x, y, z) = (x_1, z_3, y_1, z_1, x_2, y_2, y_3, x_3, z_2).$$

- Visually we depict an orbit like this:



Solution: representing points with orbits



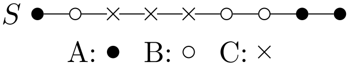
- The tuple (x, y, z) is feasible iff in the corresponding orbit x_1 is the smallest of all 1's, y_2 of all 2's and z_3 of all 3's;
- The symmetry of configuration means that switching coordinates of a given point does not change its "weight";
- Switching coordinates for an orbit means that we fix colors and then permute numbers inside each color (total of $3!^3$ permutations);
- Now, to obtain a bound, we can calculate, how many permutations, for a fixed colors, correspond to feasible tuples.

Solution for the symmetric case: controlling the orbits

Lemma (Orbits)

Let (A, B, C) be a subdivision of the set $\mathbf{9}$ into three nonintersecting subsets of size three each, and let a_1, a_2, a_3; b_1, b_2, b_3 and c_1, c_2 and c_3 be enumerations of A, B and C correspondingly. We say that such triple of enumerations is feasible if $a_1 < b_1, c_1$, $b_2 < a_2, c_2$ and $c_3 < a_1, b_1$. Then, for a fixed subdivision, the maximal number of feasible triples is 24.

Basically, we want to show that this subdivision, which corresponds to the optimal solution, is the best one:



Solution for the symmetric case: controlling the orbits

Nothing better than the case study.

○—○—•—•—•—• $\ll$ ○—○—×—×—×—○ $\mathbf{n}(S) = 12$

○—×—○—○—×—× $\ll$ ×—○—○—○—×—× $\mathbf{n}(S) = 12$

○—×—×—×—○—○

$$\begin{array}{cccccc} 2 & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \mathbf{n}(S) = 12$$

○—×—×—○—×—○

$$\begin{array}{cccccc} 2 & \cdot & \cdot & 1 & \cdot & 3 \\ 2 & \multicolumn{2}{c}{\underbrace{3}} & 3 & \cdot & 1 \\ 1 & \cdot & \cdot & 2 & 2 & 3 \end{array} \qquad \left.\begin{array}{r} :6 \\ :4 \\ :2 \end{array}\right\} \mathbf{n}(S) = 12$$

○—×—○—×—×—○

$$\begin{array}{cccccc} 2 & \cdot & 1 & \cdot & \cdot & 3 \\ 2 & 3 & 3 & \cdot & \cdot & 1 \\ 1 & \cdot & 2 & \multicolumn{2}{c}{\underbrace{2}} & 3 \end{array} \qquad \left.\begin{array}{r} :6 \\ :2 \\ :4 \end{array}\right\} \mathbf{n}(S) = 12$$

○—×—○—×—○—×

$$\begin{array}{cccccc} 2 & 3 & \cdot & \cdot & \cdot & \cdot \\ 2 & \cdot & 1 & 3 & 3 & \cdot \\ 2 & 3 & 2 & \cdot & 3 & \cdot \\ 1 & 1 & 2 & 3 & 3 & 2 \\ 1 & 3 & 3 & 1 & 2 & 2 \end{array} \qquad \left.\begin{array}{r} :4 \\ :2 \\ :2 \\ :1 \\ :1 \end{array}\right\} \mathbf{n}(S) = 10$$

○—×—×—○—○—×

$$\begin{array}{cccccc} 2 & \multicolumn{2}{c}{\underbrace{3}} & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & 2 \end{array} \qquad \left.\begin{array}{r} :8 \\ :4 \end{array}\right\} \mathbf{n}(S) = 12$$

All in all.

1. Basically, for now we can only prove a special case of a partial problem. On the other hand, the problem seem to be interesting on it own, even outside of the extremal lattice context.
2. The functional-analytic foundations of this approach should be strengthened. In particular, it would be nice to prove that the space of the configurations is compact, in some reasonable sense.
3. The main technical barrier towards the upper bound on the symmetric case is the lemma about orbits. It is not clear how to solve it in general, however it neatly embraces the combinatorial core of the problem, and hopefully, can be tackled.
4. For k's larger than four, the indications of optimality of the tentative solution are mixed. In particular, it seems that the bound, arising from the lemma about orbits, will be higher then the corresponding volume. It is thus interesting to try to design an alternative tentative configuration for $k = 4$.

Like that, but an extremal lattice