Projectable ℓ-groups and algebras of logic: Categorical and algebraic connections

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Abstract

P.F. Conrad and other authors launched a general program for the investigation of lattice-ordered groups, aimed at elucidating some order-theoretic properties of these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups. This approach can be naturally extended to residuated lattices and their convex subalgebras. In this broader perspective, we revisit the Galatos-Tsinakis categorical equivalence between integral generalized MV algebras and negative cones of ℓ-groups with a nucleus, showing that it restricts to an equivalence of the full subcategories whose objects are the projectable members of these classes. Upon recalling that projectable integral generalized MV algebras and negative cones of projectable ℓ-groups can be endowed with a positive Gödel implication, and turned into varieties by including this implication in their signature, we prove that there is an adjunction between the categories whose objects are the members of these varieties and whose morphisms are required to preserve implications.

Keywords: Residuated lattices, Lattice-ordered groups, GMV-algebras, Gödel algebras

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1. Introduction

In the 1960's, P.F. Conrad launched a general program for the investigation of lattice-ordered groups ([10], [11], [12], [13]), aimed at capturing relevant information about these algebras by inquiring into the structure of their lattices of convex ℓ-subgroups (as opposed to convex *normal* ℓ-subgroups, which had traditionally received greater attention in that they bijectively correspond to congruences). The chief idea behind this program is a working hypothesis to the effect that many significant properties of ℓ-groups are, in essence, either purely lattice-theoretical, or at least such that the underlying group structure does not play a predominant role. A class of ℓ-groups that is known to be characterized purely in terms of its order structure is the class of *projectable* ℓ-groups — namely, ℓ-groups in which every principal polar is a cardinal summand (see definition on page 11). Projectable ℓ-groups are first-class citizens in the theory of lattice-ordered groups: recall, for example, that every representable ℓ-group can be embedded into a member of this class [8] and that conditionally σ-complete ℓ-groups are projectable. Further examples arise in functional analysis, namely, vector lattices with the principal projection property [22]. One of the present authors has established that an ℓ-group is projectable iff each one of its intervals is a Stone lattice; as a consequence, projectability is preserved under lattice isomorphisms. Also, the negative cone of an ℓ-group is projectable iff its lattice reduct can be endowed with a positive Gödel implication ([24], [25], [26]).

While Conrad's program led to remarkable outcomes in its original domain of application (for a survey, see [1]), a natural continuation of such consists in extending it to *residuated lattices* ([15], [23]), generalizations of ℓ-groups that also include MV algebras, Heyting algebras, and several other classes of algebras of prime importance for mathematical logic. Here, the principal objects of research become the lattices of convex subalgebras (in the integral case, the lattices of multiplicative filters). Some detailed investigations along these lines have been carried out in recent years [7]; refer to [17] for further extensions of the Conrad program. One of the results obtained so far within this extended Conrad's program [21] is a characterization of projectability for integral and distributive residuated lattices satisfying the quasiequation

$$x \vee y \approx 1 \to xy \approx x \wedge y,$$

which closely matches the aforementioned description of projectable ℓ-groups. The last three authors of the present article have indeed shown that a member of this class is projectable iff the order dual of each interval $[a, 1]$ is a Stone lattice.

In general, for integral and distributive residuated lattices, admitting a positive Gödel implication is a stronger condition than being projectable [21, Example 15], although it is equivalent in some especially well-behaved cases. A case in point is given by *integral GMV algebras (IGMV algebras)* [16], simultaneous generalizations of MV algebras to the unbounded and noncommutative case. IGMV algebras, to within isomorphism, can be viewed as nucleus retractions of negative cones of ℓ-groups — actually, it was shown in [16] that the categories of IGMV algebras and negative cones of ℓ-groups with a nucleus are equivalent. It is then natural to conjecture that such an equivalence restricts to an equivalence of the subcategories whose objects are the projectable members of these classes of algebras, and perhaps that we can take advantage of the previously cited lattice-theoretical description of projectable IGMV algebras to establish this result. The main aim of this paper is to investigate the extent to which this conjecture is correct.

Our paper is structured as follows. Section 2, in which we go over some preliminary notions needed in the sequel, exceeds in size the average preliminaries section to be found in comparable papers, because the topics we address are rather multi-faceted — including as they do relatively pseudo-complemented lattices, residuated lattices and their structure theory, projectable residuated lattices, and IGMV algebras. Although this may be to some extent unfortunate, we deem it appropriate to review all these different aspects in some detail, to make the paper as self-contained as possible. In Section 3, we show that an analogue of the Galatos-Tsinakis equivalence result can be reproduced in our setting:

Theorem A (see Theorem 14)**.** *The categories of projectable IGMV algebras and of negative cones of projectable ℓ-groups with a nucleus are equivalent.*

A crucial step in establishing Theorem A is showing that any projectable IGMV algebra can be represented as a nucleus retract of the negative cone of some projectable ℓ-group. In the same section, we also introduce *Gödel GMV algebras* as expansions of projectable IGMV algebras by a binary term that realizes a positive Gödel implication in every such algebra; in light of the above, Gödel GMV algebras and projectable IGMV algebras amount to essentially the same thing. Similarly, *Gödel negative cones* are those Gödel GMV algebras whose RL reducts are negative cones of ℓ-groups. Including the Gödel implication in the signature enables us to view the abovementioned classes of algebras as *varieties* in the expanded type, with all the familiar benefits that result in similar cases. In Section 4, we point out the exact relationship between these notions:

Theorem B (see Theorem 28)**.** *There is an adjunction between the categories whose objects are, respectively, Gödel GMV algebras and Gödel negative cones with a retraction and a dense nucleus on the image of the retraction.*

2. Preliminaries

2.1. Pseudo-complemented and relatively pseudo-complemented lattices

A *pseudo-complemented lattice* is an algebra $\mathbf{L} = (L, \wedge, \vee, \neg, \top, \bot)$ of signature $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, \top, \bot)$ is a bounded lattice and for all $a \in L$, $\neg a = \max\{x : a \wedge x = \bot\}$. We refer to $\neg a$ as the *pseudo-complement* of a. Pseudo-complemented lattices need not be distributive, but we will henceforth assume that all lattices under consideration are such. The map $\neg : L \to L$ is a self-adjoint order-reversing map, while the map sending a to its double pseudo-complement $\neg\neg a$ is a meet-preserving closure operator on L. By a classic result due to Glivenko, the image of this closure operator is a Boolean algebra $\mathbf{B_L}$ with least element $\bot$ and largest element $\top$. Any existing meets in $\mathbf{B_L}$ coincide with those in $\mathbf{L}$; the complement of a in $\mathbf{B_L}$ is precisely $\neg a$, whereas, for any pair of elements a, b of $\mathbf{B_L}$ — also referred to as closed elements of L,

$$a \vee^{\mathbf{B_L}} b = \neg(\neg a \wedge \neg b).$$

A pseudo-complemented lattice $\mathbf{L}$ is called a *Stonean lattice* if for all $a \in L$, $\neg a \vee \neg\neg a = \top$. It can be easily seen that $\mathbf{L}$ is a Stonean lattice if and only if $\mathbf{B_L}$ is a sublattice of $\mathbf{L}$. Thus, in this case $\mathbf{B_L}$ coincides with the Boolean algebra of complemented elements of $\mathbf{L}$.

A *relatively pseudo-complemented lattice* is an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \top)$ of signature $(2, 2, 2, 0)$ such that $(A, \wedge, \vee, \top)$ is a distributive lattice with top element $\top$ and for all $a, b, c \in A$, $a \wedge b \leq c$ iff $b \leq a \to c$. Given $a, b \in A$, thus, $a \to b$ is the *relative pseudo-complement* of a with respect to b, namely, the greatest x such that $a \wedge x \leq b$. A *Heyting algebra* is an algebra $\mathbf{A} = (A, \wedge, \vee, \to, \top, \bot)$ of signature $(2, 2, 2, 0, 0)$ such that $(A, \wedge, \vee, \to, \top)$ is a relatively pseudo-complemented lattice and $\bot$ is a bottom element with respect to the lattice ordering of **A**. Observe that the $(\wedge, \vee, \neg, \top, \bot)$-term reduct of a Heyting algebra, with $\neg a = a \to \bot$, is, in particular, a pseudo-complemented lattice.

A *(positive) Gödel algebra* is a Heyting algebra (relatively pseudo-complemented lattice) satisfying the equation $(x \to y) \vee (y \to x) \approx \top$. It is important to recall that each interval $[b, \top]$ in a positive Gödel algebra can be made into a Stonean lattice by letting $\neg_b x = x \to b$ for all $x \in [b, \top]$.

Gödel algebras play a prominent role in algebraic logic because they are the equivalent variety semantics of *Gödel logic* (also known as *Dummett's logic*, or *Dummett's* **LC**), which is both an intermediate logic (i.e. an extension of intuitionistic logic) and a fuzzy logic. As an intermediate logic, it stands out for its being sound and complete with respect to linearly ordered Kripke models, and as such it received considerable attention. **LC** has been widely investigated also within the community of mathematical fuzzy logic — it was observed early on that the variety of Gödel algebras is generated by the algebra

$$([0,1], \wedge, \vee, \to, 1, 0),$$

where $\wedge$ and $\vee$ are the minimum T-norm and the maximum T-conorm respectively, while $\to$, the residual[1] of $\wedge$, behaves as follows for all $a, b \in [0,1]$:

$$a \to b = \begin{cases} 1 & \text{if } a \leq b, \\ b & \text{otherwise.} \end{cases}$$

In fact, every bounded chain admits a unique Gödel implication, given by the above case-splitting definition.

In any algebraic distributive lattice **L**, for all $a, b \in L$, the relative pseudo-complement of a with respect to b exists and is given by

$$a \to b = \bigvee \{x \in L : a \wedge x \leq b\}.$$

As a matter of fact, **L** has a bottom element and so it is a Heyting algebra. Moreover, we recall that

Lemma 1 ([3, Chapter IX, Theorem 8])**.** *If* **L** *is a Heyting algebra, every interval* $[b, a]$ *in* **L***, with* $b \leq a$*, is pseudo-complemented and, for all* $c \in$

[1]For a definition of residual, see below.

$[b,a]$, *the pseudo-complement and the double pseudo-complement of* c *are respectively given by:*

$$\neg c = (c \to b) \wedge a,$$
$$\neg\neg c = ((c \to b) \to b) \wedge a.$$

Thus, every interval in an algebraic distributive lattice is pseudo-complemented. In particular, if $a = \top$, $\uparrow b$ is itself an algebraic distributive lattice and the compact elements in $\uparrow b$ are exactly those of the form $b \vee c$, for c a compact element of L. Hereafter, by $K(\mathbf{L})$ we denote the set of all compact elements of the lattice $\mathbf{L}$.

An algebraic distributive lattice $\mathbf{L}$ is called *compactly Stonean* if it satisfies $\neg c \vee \neg\neg c = \top$, for all $c \in K(\mathbf{L})$. Observe that a compactly Stonean lattice need not be Stonean. In view of Lemma 1, $\uparrow b$ is compactly Stonean iff, for all $c \in K(\mathbf{L})$, $(c \to b) \vee ((c \to b) \to b) = \top$. It is shown in [21, Proposition 19] that:

Lemma 2. *Let* $\mathbf{L}$ *be an algebraic distributive lattice whose compact elements form a sublattice* $K(\mathbf{L})$ *of* $\mathbf{L}$*. The conditions below are equivalent:*

(1) for all $b \in L$, $\uparrow b$ *is compactly Stonean,*

(2) for all $b \in L$ *and for all* $c \in K(\mathbf{L})$, $(c \to b) \vee ((c \to b) \to b) = \top$

and imply the mutually equivalent conditions

(3) for all $c, b \in K(\mathbf{L})$, $(c \to b) \vee ((c \to b) \to b) = \top$,

(4) for all $a, b \in K(\mathbf{L})$, *with* $b \leq a$, $[b,a] \cap K(\mathbf{L})$ *is a Stonean lattice.*

The next result is straightforward and in the folklore of the subject.

Proposition 3. *Let* $\mathbf{L}$ *and* $\mathbf{M}$ *be isomorphic algebraic and distributive lattices such that* $K(\mathbf{L})$ *and* $K(\mathbf{M})$ *are subuniverses of* $\mathbf{L}$ *and* $\mathbf{M}$*, respectively. Suppose* $\varphi : \mathbf{L} \to \mathbf{M}$ *is an isomorphism. Then:*

(1) φ *preserves pseudo-complements.*

(2) $\mathbf{L}$ *is compactly Stonean iff* $\mathbf{M}$ *is such.*

(3) For $a \in L$, $\neg a$ *is complemented iff* $\varphi(\neg a) = \neg\varphi(a)$ *is complemented.*

2.2. Residuated lattices

We refer the reader to [4], [20], [23] or [15] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

A binary operation $\cdot$ on a partially ordered set $\mathbf{A} = (A, \leq)$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on A such that for all $a, b, c \in A$,

$$a \cdot b \leq c \quad \text{iff} \quad a \leq c/b \quad \text{iff} \quad b \leq a\backslash c. \tag{Res}$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$, if present.

The residuals may be viewed as generalized division operations. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element 1 and the partial order $\leq$ is a lattice order. In this case, we add the monoid unit and the lattice operation symbols to the similarity type and refer to the resulting structure $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ as a *residuated lattice*. The class of residuated lattices forms a variety (see e.g. [23, Proposition 4.5]) that we denote throughout this paper by $\mathcal{RL}$. We adopt the convention that when a class is denoted by a string of calligraphic letters, then the members of that class will be referred to by the corresponding string of Roman letters. Thus, for example, an RL is a residuated lattice.

A subvariety of $\mathcal{RL}$ of particular interest is the variety $\mathcal{CRL}$ of commutative residuated lattices, which satisfies the equation $xy \approx yx$, and hence the equation $x\backslash y \approx y/x$. We always think of this variety as a subvariety of $\mathcal{RL}$, but we slightly abuse notation by listing only one occurrence of the operation $\backslash$ in describing their members. Given an RL $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$, an element $a \in A$ is said to be *integral* if $1/a = 1 = a\backslash 1$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral. We denote by $\mathcal{IRL}$ the variety of all integral RLs. Relatively pseudo-complemented lattices are term equivalent to the subvariety $\mathcal{RPCL}$ of $\mathcal{IRL}$ that is axiomatized relative to $\mathcal{IRL}$ by the identity $xy \approx x \wedge y$. Clearly, $\mathcal{RPCL}$ is also a subvariety of $\mathcal{CRL}$.

An element $a \in A$ is said to be *invertible* if $(1/a)a = 1 = a(a\backslash 1)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case

$1/a = a^{-1} = a\backslash 1$. The RLs in which every element is invertible are precisely the ℓ-groups. Perhaps a word of caution is appropriate here. An *ℓ-group* is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, {}^{-1}, e)$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, {}^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety $\mathcal{LG}$ of $\mathcal{RL}$ defined by the equations $(1/x)x \approx 1 \approx x(x\backslash 1)$; the term equivalence is given by $x^{-1} = 1/x$ and $x/y = xy^{-1}, x\backslash y = x^{-1}y$. Throughout this paper, the members of this subvariety will be referred to as ℓ-groups *simpliciter*. Negative cones of ℓ-groups are RLs as well. If $\mathbf{G}$ is an ℓ-group, in fact, $G^- = \{x \in G : x \leq 1\}$ is the universe of an RL $\mathbf{G}^-$ such that, for all $a, b \in G^-$, $a\backslash^{\mathbf{G}^-} b = a\backslash^{\mathbf{G}} b \wedge 1$, and similarly for right residuals.

GMV algebras [16], to which we shall soon revert in much greater detail, are simultaneous generalizations of *MV algebras* [9] to the noncommutative, unbounded and nonintegral case. The variety $\mathcal{GMV}$ of GMV algebras is axiomatized relative to $\mathcal{RL}$ by the equations

E1. $x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x$.

It is essential to note that GMV algebras have distributive lattice reducts [16, Lemma 2.9]. Since this property is shared by all the RLs we deal with in the following, *hereafter we assume that any RL we consider is distributive as a lattice*. The variety $\mathcal{IGMV}$ of integral GMV algebras, of course, is axiomatized relative to $\mathcal{IRL}$ by the equation E1, which in this context simplifies to

E2. $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$.

The class $\mathcal{LG}^-$ of negative cones of ℓ-groups is a subvariety of $\mathcal{IGMV}$, axiomatized relative to $\mathcal{IGMV}$ [2, Theorem 6.2] by the equations

E3. $x\backslash xy \approx y \approx yx/x$.

2.3. Nuclei on residuated lattices

A *nucleus* on an RL $\mathbf{A}$ is a closure operator γ on $\mathbf{A}$ satisfying one of the following equivalent conditions for all $a, b \in A$:

$$\gamma(a)\gamma(b) \leq \gamma(ab),$$
$$\gamma(\gamma(a)\gamma(b)) = \gamma(ab).$$

If $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is an RL and γ is a nucleus on $\mathbf{A}$, the image A_γ of γ can be endowed with an RL structure as follows:

$$\mathbf{A}_\gamma = (A_\gamma, \wedge, \vee_\gamma, \cdot_\gamma, \backslash, /, \gamma(1)),$$

where

$$\gamma(a) \vee_\gamma \gamma(b) = \gamma(a \vee b),$$
$$\gamma(a) \cdot_\gamma \gamma(b) = \gamma(a \cdot b).$$

$\mathbf{A}_\gamma$ is called a *nucleus retract* of $\mathbf{A}$.

Nuclei on GMV algebras have a few special properties. In fact, if $\mathbf{A}_\gamma$ is a nucleus retract of an (integral) GMV algebra, then $\vee_\gamma = \vee$, $\gamma(1) = 1$ and

$$\mathbf{A}_\gamma = (A_\gamma, \wedge, \vee, \cdot_\gamma, \backslash, /, 1)$$

is an (integral) GMV algebra in its own right. In particular, it follows on the one hand that nuclei on GMV algebras are lattice homomorphisms, and on the other, that nucleus retracts of negative cones of ℓ-groups (*qua* instances of IGMV algebras) are themselves IGMV algebras.

2.4. Filters in integral residuated lattices

Let $\mathbf{A}$ be an RL. A *multiplicative filter* F of $\mathbf{A}$ is a filter of its lattice reduct that is closed under multiplication. A subset $X \subseteq A$ (not necessarily a filter) is *normal* provided that for all $b \in X$ and $a \in A$, $\rho_a(b) = (ab/a) \wedge 1$ and $\lambda_a(b) = (a \backslash ba) \wedge 1$ are in X. The map sending x to $\rho_a(x)$ (respectively, to $\lambda_a(x)$) is called a *right* (respectively, *left*) *conjugation map*, and $\rho_a(x)$ ($\lambda_a(x)$) is said to be the *right* (*left*) *conjugate* of x by a.

Observe that $\mathcal{RL}$ is both congruence permutable (witness the term $[z \vee (z/y)x] \wedge [x \vee (x/y)z]$) and 1-regular (witness the terms $x \backslash y \wedge 1, y \backslash x \wedge 1$), and recall that any variety which is congruence permutable and 1-regular is, in particular, *ideal determined*: the lattice of congruence relations and the lattice of ideals (in the sense of [18]) of any algebra in the variety are isomorphic. It is proved in [4] (see also [15]) that for any RL $\mathbf{A}$, ideals of $\mathbf{A}$ coincide with convex normal subalgebras of $\mathbf{A}$. If $\mathbf{A}$ is integral, these coincide, in turn, with normal multiplicative filters of $\mathbf{A}$.

Now, let $\mathbf{A}$ be an IRL. If $X \subseteq A$, we denote by $\uparrow_{\mathbf{A}}(X)$ (respectively, $\langle X \rangle_{\mathbf{A}}$, $N_{\mathbf{A}}(X)$) the lattice filter (resp. multiplicative filter, normal multiplicative filter) generated in $\mathbf{A}$ by X. Subscripts will only be dropped when $\mathbf{A}$ is understood; on the other hand, braces will be invariably omitted if $X = \{a\}$ is a singleton. $\mathrm{F}(\mathbf{A})$, $\mathrm{MF}(\mathbf{A})$, $\mathrm{NF}(\mathbf{A})$ will respectively refer to the lattices of lattice filters, multiplicative filters and normal multiplicative filters (hereafter shortened to normal filters) of $\mathbf{A}$. With a mild abuse of notation, the same labels will sometimes be employed for the universes of such lattices. We set:

$$F \vee^L G = \uparrow(F \cup G),$$
$$F \vee^M G = \langle F \cup G \rangle,$$
$$F \vee^N G = N(F \cup G).$$

However, since the focus of the present paper is on multiplicative filters, we will often write $F \vee G$ for $F \vee^M G$. $\mathrm{F}(\mathbf{A})$, $\mathrm{MF}(\mathbf{A})$, and $\mathrm{NF}(\mathbf{A})$ are algebraic and distributive (hence relatively pseudo-complemented) lattices; the result for $\mathrm{MF}(\mathbf{A})$ is proved in [7]. We also recall:

Lemma 4. $\langle X \rangle = \{a : (b_1 \cdots b_k)^n \leq a,$ *for some* $b_1, \ldots, b_k \in X$ *and* $n \in \mathbb{N}\}$.

An *iterated conjugation map* is a composition $\gamma = \gamma_1 \circ \cdots \circ \gamma_n$, where each γ_i is a right conjugate or a left conjugate by an element $a_i \in A$. If $X \subseteq A$, we denote by Γ the set of all iterated conjugation maps on $\mathbf{A}$, and by $\widehat{X}$ the submonoid of the corresponding reduct of $\mathbf{A}$ generated by the set $\{\gamma(a) : a \in X, \gamma \in \Gamma\}$. With this notation at hand, we recall the following result from [4] (see also [23, Proposition 4.24]):

Lemma 5. $N(X) = \{a : b \leq a,$ *for some* $b \in \widehat{X}\}$.

We now introduce a technique for defining multiplicative filters out of arbitrary subsets of the universe of a given IRL. Given $X \subseteq A$, the *polar* $X^\perp$ of X is the set

$$\{y \in A : x \vee y = 1 \text{ for every } x \in X\}.$$

Again, whenever $X = \{a\}$ is a singleton, we will shorten $\{a\}^\perp$ to $a^\perp$ and call the latter set a *principal polar*. In case $\mathbf{A}$ is distributive, we have that [21, Lemma 8 and Corollary 9]:

Lemma 6. *For all* $X \subseteq A$, $X^\perp \in \mathrm{MF}(\mathbf{A})$. *Moreover,* $X^\perp$ *is the pseudo-complement of* $\uparrow X$ *in* $\mathrm{F}(\mathbf{A})$ *(respectively, the pseudo-complement of* $\langle X \rangle$ *in* $\mathrm{MF}(\mathbf{A})$*).*

On the other hand, given an arbitrary $X \subseteq A$, $X^\perp$ need not be a *normal* filter of $\mathbf{A}$; if it is, then it is the pseudo-complement of $N(X)$ in $\mathrm{NF}(\mathbf{A})$.

Lemma 7. $(\uparrow a)^\perp = \langle a \rangle^\perp = a^\perp$.

Proof. Use Lemmas 4 and 5 above. □

2.5. Projectable residuated lattices

An ℓ-group $\mathbf{A}$ is the *internal cardinal product* of its ℓ-subgroups $\mathbf{B}$ and $\mathbf{C}$ (in symbols, $\mathbf{A} = \mathbf{B} \boxplus \mathbf{C}$) if every $a \in A$ can be written uniquely as a product bc, for some $b \in B$ and some $c \in C$, and moreover, $a_1 = b_1 c_1 \leq^{\mathbf{A}} b_2 c_2 = a_2$ iff $b_1 \leq^{\mathbf{B}} b_2$ and $c_1 \leq^{\mathbf{C}} c_2$. An ℓ-group $\mathbf{A}$ is *projectable* whenever for all $a \in A$, $\mathbf{A} = a^\nabla \boxplus a^{\nabla\nabla}$, where in the present context $a^\nabla = \{b \in A : |a| \wedge |b| = 1\}$ and $|a| = a \vee a^{-1}$. As proved in [24] and [25], projectable ℓ-groups coincide with ℓ-groups in which all closed intervals form a Stonean lattice, and hence they admit a Gödel implication. This result highlights that projectability is a property of ℓ-groups that is entirely determined by their order structure. To get further insight into this, recall indeed that, given an ℓ-group $\mathbf{A}$: (1) principal polars are convex ℓ-subgroups of $\mathbf{A}$; (2) projectability is equivalent to the property that for all $a \in A$, $\mathbf{A} = a^\nabla \vee^M a^{\nabla\nabla}$; (3) the lattice of convex ℓ-subgroups of $\mathbf{A}$ is isomorphic to the lattice of convex submonoids of its negative cone $\mathbf{A}^-$, and in particular $a^\nabla \cap A^- = \{b \in A : a \vee b = 1\}$; (4) the crucial observation here is that for all $a \in A$, $\mathbf{A} = a^\nabla \vee a^{\nabla\nabla}$, where the join is taken in the lattice of filters of the negative cone of $\mathbf{A}$; this makes clear that projectability is an order-theoretic property. (See [26] for a lengthier discussion of these aspects.)

As already observed, negative cones of ℓ-groups make instances of IRLs. Moreover, negative cones satisfy the quasi-equation

$$x \vee y \approx 1 \to xy \approx x \wedge y. \tag{1}$$

In [21], the above lattice-theoretic characterization of projectable ℓ-groups and their negative cones has been extended to the class $\mathcal{A}$ of IRLs satisfying that quasi-equation. Throughout this subsection, unless otherwise specified, we will assume that $\mathbf{A}$ is a member of $\mathcal{A}$.

An IRL $\mathbf{A}$ is called *projectable* if for all $a \in A$, it can be written as a cardinal product $\mathbf{A} = a^\perp \boxplus a^{\perp\perp}$. For members of $\mathcal{A}$, projectability is a lattice-theoretic property, in the sense that it can be "captured" by the filter lattice of the underlying lattice-structure.

Lemma 8. *If* $\mathbf{A}$ *is projectable, then:*

(1) $a^\perp \in \mathrm{NF}(\mathbf{A})$*,*

(2) $\mathbf{A} = a^\perp \vee^L a^{\perp\perp} = a^\perp \vee^M a^{\perp\perp}$*.*

Lemma 2 can be put to good use by applying it to the lattice $\mathrm{F}(\mathbf{A})$ of lattice filters of our $\mathbf{A} \in \mathcal{A}$. In fact, if $\mathbf{A}$ is projectable, then the compact elements of the lattice $\mathrm{F}(\mathbf{A})$ of the lattice filters of $\mathbf{A}$ are its principal filters, whereby $\mathrm{F}(\mathbf{A})$ is compactly Stonean. This implies that each interval $[\{1\}, \uparrow a]$ in the sublattice of principal lattice filters of $\mathbf{A}$ is a Stonean lattice. In light of the order reversing isomorphism between the lattice reduct of $\mathbf{A}$ and the sublattice of principal filters in $\mathrm{F}(\mathbf{A})$, then, for all $a \in H$ the order dual of each interval $[a, 1]$ is a Stonean lattice. In sum:

Theorem 9 ([21, Theorem 20])**.** $\mathbf{A}$ *is projectable iff the order dual of each interval* $[a, 1]$*, for* $a \in A$*, is a Stonean lattice.*

In particular, if $\mathbf{A}$ is an IGMV algebra, we get something more. Every member x of any such interval is a fixpoint of the mapping $f_a(x) = a/(x\backslash a)$, whence the interval is self-dual in the order-theoretic sense. It follows that every interval $[a, 1]$, and therefore any arbitrary interval $[a, b]$ is a Stonean lattice (see [3, § 8.7, Theorem 13]). Thus, following [3, Theorem 10, p. 176], $(A, \wedge, \vee)$ is a relative Stonean lattice and, as such, it can be expanded to a relatively pseudo-complemented lattice, actually a positive Gödel algebra. In conclusion, we have:

Lemma 10. *For an IGMV algebra* $\mathbf{A}$*, the following are equivalent:*

(1) $\mathbf{A}$ *is projectable.*

(2) The lattice $(A, \wedge, \vee)$ *can be expanded to a relatively pseudo-complemented lattice, actually a positive Gödel algebra.*

2.6. Integral GMV algebras

We recalled in Section 2.3 that nucleus retracts of negative cones of ℓ-groups are IGMV algebras. In this section, we sketch the construction in [16, § 3] by means of which Galatos and Tsinakis establish the converse, namely that every IGMV algebra is a nucleus retract of the negative cone of an ℓ-group. This representation theorem is subsequently lifted [16, § 4] to a full-fledged categorical equivalence between the categories of IGMV algebras and of negative cones of ℓ-groups endowed with a nucleus. This result will be briefly summarized as well.

The first part of the construction relies on an idea by Bosbach, aimed at identifying the purely implicational subreducts of negative cones of ℓ-groups ([5], [6]). A *cone algebra* is an algebra $\mathbf{C} = (C, \backslash, /, 1)$, of type $(2, 2, 0)$, that satisfies the identities:

C1. $(x\backslash y)\backslash(x\backslash z) \approx (y\backslash x)\backslash(y\backslash z)$

C2. $1\backslash x \approx x$

C3. $x\backslash(y/z) \approx (x\backslash y)/z$

C4. $x\backslash x \approx 1$

as well as their mirror images (in the RL sense). The variety of cone algebras will be sometimes referred to as $\mathcal{CA}$. It is easily seen that the $(2, 2, 0)$-reducts of IGMV algebras are cone algebras. Bosbach shows that the converse holds true too. More precisely:

Proposition 11 ([6]). *Every cone algebra can be embedded into the* $(2, 2, 0)$-*reduct of an appropriate member of* $\mathcal{LG}^-$.

Sketch of the proof. The target negative cone is obtained as a union of an ascending chain $\{\mathbf{C}_n : n < \omega\}$ of cone algebras, each of which is a subalgebra of its successor. Products in the target algebra are constructed stepwise, in such a way that each $\mathbf{C}_{n+1}$ contains products of members of $\mathbf{C}_n$, until all products are finally available in the directed union of the $\mathbf{C}_i$'s.

In greater detail, we proceed as follows. Given a cone algebra $\mathbf{C}$ and elements $(a, b), (c, d)$ in C^2, le

$$(a, b)\backslash\!\!\backslash(c, d) = (b\backslash(a\backslash c), ((a\backslash c)\backslash b)\backslash((c\backslash a)\backslash d)),$$
$$(d, c)/\!\!/(b, a) = ((d/(a/c))/(b/(c/a)), (c/a)/b).$$

The rationale for this definition is given by the fact that $\mathcal{LG}^-$ satisfies the identity

E5. $xy\backslash zw \approx (y\backslash(x\backslash z)) \cdot (((x\backslash z)\backslash y)\backslash((z\backslash x)\backslash w))$

and its mirror image, whence the Cartesian product operation, so to speak, acts as an *ersatz* for the RL product and $\backslash\!\!\backslash, /\!\!/$ can be viewed as residuals of sorts. Now, the relation

$$\Theta = \left\{((a,b),(c,d)) : (a,b)\backslash\!\!\backslash(c,d) = (1,1) = (c,d)/\!\!/(a,b)\right\}$$

is a congruence on $\mathbf{C}^2$, and

$$s(\mathbf{C}) = \mathbf{C}^2/\Theta$$

is a cone algebra containing $\mathbf{C}$ as a subalgebra, via the embedding $\varphi(a) = [(a,1)]_\Theta$. To attain our target negative cone, we run this construction over and over again, letting $\mathbf{C}_0 = \mathbf{C}$ and $\mathbf{C}_{n+1} = s(\mathbf{C}_n)$. In this way, in each $\mathbf{C}_i$ E4 is satisfied by the elements of C_j, $j \leq i-1$. The directed union $\overline{\mathbf{C}} = \bigcup\{\mathbf{C}_n : n < \omega\}$ is a cone algebra that still contains $\mathbf{C}$ as a subalgebra. Moreover, it is the $(2,2,0)$-reduct of the negative cone

$$\widehat{\mathbf{C}} = \left(\overline{C}, \wedge, \vee, \cdot, \backslash^{\overline{\mathbf{C}}}, /^{\overline{\mathbf{C}}}, 1^{\overline{\mathbf{C}}}\right),$$

where $ab = [(a,b)]_\Theta$, $a \vee b = a/^{\overline{\mathbf{C}}}(b\backslash^{\overline{\mathbf{C}}}a)$ and $a \wedge b = (a/^{\overline{\mathbf{C}}}b)b$. □

We make a note of the fundamental fact that every element of $\overline{C}$ can be written as a product of members of C, and proceed to outline the proof of the representation theorem for IGMV algebras. Hereafter, we find convenient to use the term *dense nucleus* for a nucleus on the negative cone $\mathbf{G}^-$ whose image G^-_γ generates $\mathbf{G}^-$ as a monoid. In what follows, we use the expression $\prod^{\mathbf{A}}_{j\leq n} x_j$ for the product $x_1 \cdot^{\mathbf{A}} \cdots \cdot^{\mathbf{A}} x_n$.

Theorem 12 ([16, Theorem 3.12])**.** *An IRL is a GMV algebra if and only if it is the retract of a dense nucleus on the negative cone of some ℓ-group.*

Sketch of the proof. We are going to prove only the forward direction. For the converse, we refer the reader to [16, Theorem 3.4]. Consider an IGMV algebra $\mathbf{A} = (A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, \cdot^{\mathbf{A}}, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, 1^{\mathbf{A}})$. The crucial observation, here, is that its implicative reduct $(A, \backslash^{\mathbf{A}}, /^{\mathbf{A}}, 1^{\mathbf{A}})$ is a cone algebra, whence by Proposition 11 it can be embedded into the corresponding reduct of a RL $\mathbf{G}^- \in \mathcal{LG}^-$ which is generated by A as a monoid. All we need for our claim to hold true is some nucleus γ that makes the nucleus retract $\mathbf{G}^-_\gamma$ isomorphic to $\mathbf{A}$. To this effect, let $a = \prod_{j \leq n}^{\mathbf{G}^-} a_j \in G^-$, where each $a_j \in A$, and define

$$\gamma(a) = \gamma\left(\prod_{j \leq n}^{\mathbf{G}^-} a_j\right) = \prod_{j \leq n}^{\mathbf{A}} a_j.$$

This map is well-defined and is actually a nucleus on $\mathbf{G}^-$. Clearly, the universe of the nucleus retract $\mathbf{G}^-_\gamma$ coincides with $\mathbf{A}$, and it can be seen that the operations in both structures coincide with one another. In particular, $a \cdot^{\mathbf{G}^-_\gamma} b = \gamma(a \cdot^{\mathbf{G}^-} b) = a \cdot^{\mathbf{A}} b$. □

The preceding representation theorem can be actually viewed as just part of a more general categorical equivalence. The categories in point are IGMV, the category whose objects are IGMV algebras and whose morphisms are RL homomorphisms, and LG^-_*, the category whose objects are expansions of negative cones by a dense nucleus γ, and whose morphisms are RL homomorphisms that preserve γ.

Theorem 13. *The categories* IGMV *and* LG^-_* *are equivalent.*

Sketch of the proof. Let $\mathbf{K}$ be an object in LG^-_*; we let $\Gamma(\mathbf{K}) = \mathbf{K}_{\gamma^{\mathbf{K}}}$. Moreover, if $f : \mathbf{K} \to \mathbf{L}$ is a morphism in LG^-_*, we define $\Gamma(f)$ as the restriction of f to $K_{\gamma^{\mathbf{K}}}$. We prove in turn each of the following items:

- Γ *is a well-defined functor.* $\Gamma(\mathbf{K})$ is an object in IGMV because nucleus retracts of negative cones of ℓ-groups are IGMV algebras (Theorem 12). It can be easily checked that $\Gamma(f)$ is a morphism in IGMV, essentially because f commutes with the nuclei $\gamma^{\mathbf{K}}, \gamma^{\mathbf{L}}$. It is immediate that Γ preserves composition of arrows and the identity morphism.

- Γ *is full.* Every object in IGMV is the Γ-image of an object in LG^-_* by Theorem 12. It takes a lot more work to show that Γ is surjective on morphisms; however, by using a variation on Cignoli and Mundici's technique of *good sequences* [9, Chapter 2], it is possible to prove that whenever we are given objects $\mathbf{K}, \mathbf{L}$ in LG^-_* and a homomorphism f from $\mathbf{K}_{\gamma^{\mathbf{K}}}$ to $\mathbf{L}_{\gamma^{\mathbf{L}}}$, there exist a unique RL homomorphism $\overline{f} : \mathbf{K} \to \mathbf{L}$ such that
$$f \circ \gamma^{\mathbf{K}} = \gamma^{\mathbf{L}} \circ \overline{f},$$
whence the claim follows.

- Γ *is faithful.* Since γ is assumed to be dense, $\mathbf{K} = \mathbf{L}$ whenever $\mathbf{K}_{\gamma^{\mathbf{K}}} = \mathbf{L}_{\gamma^{\mathbf{L}}}$ and, for $f, g : \mathbf{K} \to \mathbf{L}$, $f = g$ in case $f \restriction K_{\gamma^{\mathbf{K}}} = g \restriction K_{\gamma^{\mathbf{K}}}$.

- Γ *is essentially surjective.* This is actually the content of Theorem 12.

This much suffices for our main claim. □

3. Projectable IGMV algebras and projectable lattice-ordered groups

The results in §§ 2.5 and 2.6 suggest a very natural conjecture to the effect that suitable analogues of Theorems 12 and 13 continue to hold for projectable IGMV algebras. More precisely, it seems plausible to surmise that such algebras — which, by virtue of Lemma 10, coincide with IGMV algebras that admit a positive Gödel implication — are nucleus retracts of negative cones of projectable ℓ-groups, and that the corresponding categories are equivalent to each other. In this section, we will see that both statements actually hold, if appropriately qualified. Namely, the equivalence between the categories of IGMV algebras and of negative cones of ℓ-groups restricts to an equivalence of the respective full subcategories whose objects are the projectable members, and whose morphisms are γ-preserving RL homomorphisms.

In greater detail, let PLG_*^- be the category whose objects are negative cones of projectable ℓ-groups equipped with a dense nucleus γ, and whose arrows are their γ-preserving RL homomorphisms; analogously, let PGMV will be the category whose objects are projectable IGMV algebras and whose arrows are their RL homomorphisms. We will prove in this section that:

Theorem 14. PLG_*^- *and* PGMV *are equivalent.*

If we want the signature of our algebras to include the Gödel implication, and our category morphisms to preserve it — turning projectable IGMV algebras and negative cones of projectable ℓ-groups into *varieties,* so as to profit from the well-known advantages yielded by this move — the exact relationship between the resulting categories is not as simple as that, although we will defer to the next section a detailed investigation of the problem.

Let $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, 1)$ be a projectable IGMV algebra. The construction of Theorem 12 vouches for the existence of an ℓ-group $\mathbf{G}$, and of a dense nucleus γ on its negative cone $\mathbf{G}^-$, such that $\mathbf{M}$ is isomorphic to $\mathbf{G}_\gamma^-$.

Lemma 15. *Let a be a member of G^- such that $a < 1$. Then, there exists $b \in G_\gamma^-$ such that $a \leq b < 1$.*

Proof. Since γ is dense, we know that for some $x_1, \ldots, x_n$ we have that $a = \prod_{j \leq n}^{\mathbf{G}^-} \gamma(x_j)$. For some k, $\gamma(x_k) < 1$ (otherwise $a = \prod_{j \leq n}^{\mathbf{G}^-} \gamma(x_j) = 1$, a contradiction). Pick such a k. Then,

$$a = \prod_{j \leq n}^{\mathbf{G}^-} \gamma(x_j) \leq \gamma(x_k) < 1. \qquad \square$$

Lemma 16. *Let $\mathbf{G}^-$ be the negative cone of an ℓ-group, and let γ be a dense nucleus on $\mathbf{G}^-$ with image $\mathbf{G}_\gamma^-$. The lattices $\mathrm{MF}(\mathbf{G}^-)$ and $\mathrm{MF}(\mathbf{G}_\gamma^-)$ of multiplicative filters of $\mathbf{G}^-$ and $\mathbf{G}_\gamma^-$, respectively, are isomorphic. The isomorphism is given by the mutually inverse maps $\varphi(F) = \langle F \rangle_{\mathbf{G}^-}$ and $\psi(H) = \gamma[H] = H \cap G_\gamma^-$.*

Proof. Let $F, H \in \mathrm{MF}(\mathbf{G}_\gamma^-)$. Now, if $\langle F \rangle_{\mathbf{G}^-} = \langle H \rangle_{\mathbf{G}^-}$ and $a \in F$, then $a \in \langle F \rangle_{\mathbf{G}^-} = \langle H \rangle_{\mathbf{G}^-}$, whence there exist $h_1, \ldots, h_n \in H$ such that $\prod_{j \leq n}^{\mathbf{G}^-} h_j \leq a$. So

$$\prod_{j \leq n}^{\mathbf{G}_\gamma^-} h_j = \gamma\Big(\prod_{j \leq n} h_j\Big) \leq \gamma(a) = a,$$

and thus $a \in H$.

For surjectivity, it suffices to show that an arbitrary multiplicative filter J of $\mathbf{G}^-$ is such that $J = \langle \gamma[J] \rangle_{\mathbf{G}^-}$. For the nontrivial direction, let $a \in J$. Since γ is dense, $a = \prod_{i \leq m}^{\mathbf{G}^-} h_i$, for some $h_1, \ldots, h_m \in G_\gamma^-$; so, in particular, $\prod_{i \leq m}^{\mathbf{G}^-} h_i \leq a$ and $a \in \langle \gamma[J] \rangle_{\mathbf{G}^-}$. Order preservation is clear. $\square$

In the next Lemma we make a note of some interesting properties of generated filters and of the mappings φ and ψ in Lemma 16. In the interests of readability, we write $F^{\perp_\gamma}$ in place of $F^{\perp_{\mathbf{G}_\gamma^-}}$, and $F^{\perp}$ in place of $F^{\perp_{\mathbf{G}^-}}$. Also, we let $\langle X \rangle_\gamma$ stand for $\langle X \rangle_{\mathbf{G}_\gamma^-}$ and $\langle X \rangle$ for $\langle X \rangle_{\mathbf{G}^-}$.

Lemma 17. *Let $\mathbf{G}^-$ be the negative cone of a projectable ℓ-group, and let $\mathbf{G}_\gamma^-$ be a nucleus retract of it, with γ a dense nucleus.*

(1) For any $a \in G^-$, $\psi(\langle a \rangle) = \langle \gamma(a) \rangle_\gamma$.

(2) For any $a \in G^-_\gamma$, $\varphi(\langle a \rangle_\gamma) = \langle a \rangle$.

(3) For any $a \in G^-_\gamma$, $\varphi(a^{\perp_\gamma}) = a^\perp$.

(4) For any $a \in G^-$, $\psi(a^\perp) = \gamma(a)^{\perp_\gamma}$.

(5) If $a \in G^-_\gamma$, $\varphi(a^{\perp_\gamma})$ is a complemented element in $\mathrm{MF}(\mathbf{G}^-)$, its complement being $(a^{\perp_\gamma})^\perp$.

Proof.
(1) For the nontrivial direction, let $x \in \psi(\langle a \rangle) = \langle a \rangle \cap G^-_\gamma$. Thus $x \geq a^n$, for some $n \in N$. It follows that $x = \gamma(x) \geq \gamma(a^n) = \gamma(a) \cdot^{\mathbf{G}^-_\gamma} \cdots \cdot^{\mathbf{G}^-_\gamma} \gamma(a)$, whence our claim follows.

(2) From (1), by applying the isomorphism φ on both sides.

(3) By Proposition 3.(1) and item (2),

$$\varphi(a^{\perp_\gamma}) = \varphi(\langle a \rangle_\gamma^{\perp_\gamma}) = \varphi(\langle a \rangle_\gamma)^\perp = \langle a \rangle^\perp = a^\perp.$$

(4) By Proposition 3.(1) and item (1),

$$\psi(a^\perp) = \psi(\langle a \rangle^\perp) = (\psi(\langle a \rangle))^{\perp_\gamma} = (\langle \gamma(a) \rangle_\gamma)^{\perp_\gamma} = \gamma(a)^{\perp_\gamma}.$$

(5) From Proposition 3.(1)–(3). □

Lemma 18. *An IRL $\mathbf{M}$ is a projectable IGMV algebra if and only if it is a retract of a dense nucleus on the negative cone $\mathbf{G}^-$ of some projectable ℓ-group.*

Proof. In view of the previous Lemma and in virtue of Theorem 12 we confine ourselves to proving the left to right direction. Let $\mathbf{M}$ be a projectable IGMV algebra. We use the construction in Theorem 12 to obtain an ℓ-group $\mathbf{G}$, and a nucleus γ on its negative cone $\mathbf{G}^-$, such that $\mathbf{M}$ is isomorphic to $\mathbf{G}^-_\gamma$. It remains to show that $\mathbf{G}$ is projectable. Now, by Lemma 10, $\mathbf{G}^-_\gamma$ is projectable, and this property is witnessed by its lattice of multiplicative filters; namely, for all $a \in G^-_\gamma$,

$$a^{\perp_\gamma} \vee a^{\perp_\gamma \perp_\gamma} = G^-_\gamma.$$

Now, recall that for our claim to hold, it suffices to show that $\mathbf{G}^-$ is projectable, namely that for all $a \in G^-$,

$$a^\perp \vee a^{\perp\perp} = G^-.$$

This much will suffice, because the map that sends convex subalgebras of an ℓ-group to convex subalgebras of its negative cone is an isomorphism. Let $a \in G^-$. Then

$$\begin{aligned}
\psi(a^\perp \vee a^{\perp\perp}) &= \psi(a^\perp) \vee \psi(a^{\perp\perp}) && \psi \text{ preserves joins} \\
&= \psi(a^\perp) \vee \psi(a^\perp)^{\perp_\gamma} && \text{Proposition 3.(1)} \\
&= \gamma(a)^{\perp_\gamma} \vee \gamma(a)^{\perp_\gamma\perp_\gamma} = G^-_\gamma, && \text{Lemma 17.(4)}
\end{aligned}$$

whence our conclusion follows given that ψ is an isomorphism. □

We now proceed to the proof of Theorem 14.

Proof of Theorem 14. Lemma 10 and Lemma 18 imply that PLG^-_* and PGMV are full subcategories of the categories LG^-_* and IGMV, respectively. So, the functor Γ in Theorem 13 restricts to a full and faithful functor from PLG^-_* to PGMV, whence our claim follows. □

Corollary 19. *The categories of projectable MV algebras and projectable unital Abelian ℓ-groups are equivalent.*

Let us remark that the object part of this equivalence was already observed in [19].

4. Introducing the categories GLG^- and GGMV

As already observed, it is natural to give an equational characterization of projectability by including in the signature the operation symbol for the Gödel implication. If so, our category morphisms should obviously preserve the additional operation, but the morphisms in both PLG^-_* and PGMV fall short of this *desideratum*. As a counterexample, consider the negative cone of the lexicographic product of $\mathbb{Z}$ by $\mathbb{Z}$, and let p be the natural projection onto $\mathbb{Z}^-$. It can be seen that $p((-1,-1) \to (-1,-2)) = p((-1,-2)) = -1$, while $p((-1,-1)) \to p((-1,-2)) = -1 \to -1 = 0$. As we shall see in this section, however, imposing this further constraint upon our arrows will downgrade the previous equivalence to an *adjunction*.

One might hope that the restriction of the functor Γ to the subcategories of PLG_*^- and PGMV, consisting of the same objects and morphisms that also respect the Gödel arrows, might lead to an adjunction between these two categories. However, it is an open problem at this time whether the restriction of Γ is well defined, namely, that the image $\Gamma(f)$ of a morphism f in GLG^- respecting the Gödel arrow respects the Gödel arrow. In fact, we conjecture that this is not the case, and the treatment below is necessary for establishing the aforementioned adjunction.

Thus, in what follows, we will deal with projectable IGMV algebras in the signature expanded by an additional binary operation symbol $\to$, which denotes the relative pseudo-complement whose existence is guaranteed by Lemma 10. To distinguish these algebras from their $\to$-free counterparts we need a special label, provided via the next definition.

Definition 20. A *Gödel GMV algebra* is an algebra $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, \to, 1)$ of type $(2, 2, 2, 2, 2, 2, 0)$ such that:

(1) $(M, \wedge, \vee, \cdot, \backslash, /, 1)$ is an IGMV algebra,

(2) $(M, \wedge, \vee, \to, 1)$ is a positive Gödel algebra.

The labels $\mathcal{GGMV}$ and $\mathcal{GLG}^-$ will henceforth stand for the varieties of Gödel GMV algebras and of Gödel negative cones (Gödel GMV algebras whose RL reducts are negative cones of ℓ-groups), respectively.

Theorem 21. *Any Gödel GMV algebra* $\mathbf{M} = (M, \wedge, \vee, \cdot, \backslash, /, \to, 1)$ *is the retract of a dense nucleus*[2] *of some Gödel negative cone.*

Proof. Using the notation of Lemma 18, the claim will follow if we can show that $\to^{\mathbf{M}}$ coincides with the relative pseudo-complement in $\mathbf{G}_\gamma^-$, whose existence is guaranteed by the fact that $\mathbf{G}^-$ is projectable. However, if $a, b \in G_\gamma^-$, $a \to^{\mathbf{M}} b$ is a closed element in that $\gamma(b) = b \leq a \to^{\mathbf{M}} b$, and closed elements form a lattice filter of $\mathbf{M}$. Since it is the largest x such that $a \wedge x \leq b$, in particular it is the largest closed element with that property. In sum,

$$a \to^{\mathbf{M}} b = \max\{\gamma(x) : a \wedge \gamma(x) \leq b\} = a \to^{\mathbf{G}_\gamma^-} b. \qquad \square$$

[2]Notice that by nucleus, here, we mean a nucleus on the RL reduct of $\mathbf{A}$; it should be pointed out that, by [16, Corollary 3.7], such nuclei also preserve meets.

The preceding proof also yields:

Corollary 22. *The* $\{\backslash, /, \rightarrow, 1\}$*-reduct of any Gödel GMV algebra is a subreduct of a Gödel negative cone.*

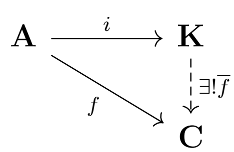
For our purposes, the following generalization (for which see e.g. [27]) of the usual concept of free algebra over a set of free generators will come in handy.

Definition 23. Let $\mathcal{K}$ and $\mathcal{K}'$ be classes of algebras of respective signatures ν and ν', with $\nu' \subset \nu$. The algebra $\mathbf{K} \in \mathcal{K}$ is a *$\mathcal{K}$-free extension* over $\mathbf{A} \in \mathcal{K}'$ in case:

(1) $\mathbf{A}$ is a ν'-subreduct of $\mathbf{K}$.

(2) The subalgebra of $\mathbf{K}$ generated by A is $\mathbf{K}$.

(3) Every homomorphism of $\mathbf{A}$ to the ν'-reduct of any $\mathbf{C} \in \mathcal{K}$ can be extended to a unique homomorphism of $\mathbf{K}$ to $\mathbf{C}$.



To make terminology less cumbersome, we will refer to the $\mathcal{K}$-free extension over $\mathbf{A}$ as "the free K over $\mathbf{A}$." Thus, for example, the $\mathcal{LG}^{-}$-free extension over $\mathbf{A} \in \mathcal{CA}$ will be described as the free negative cone over $\mathbf{A}$.

Lemma 24. *Let* $\mathbf{M}$ *be a Gödel GMV algebra, and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be its* $\{\backslash, /, \rightarrow, 1\}$*-reduct*[3] *and its* $\{\backslash, /, 1\}$*-reduct, respectively. Then:*

(1) The free Gödel negative cone $\mathbf{K}$ *over* $\mathbf{A}$ *exists.*

(2) The RL subreduct $\mathbf{L}$ *generated by* B *in the* $\{\rightarrow\}$*-free reduct of* $\mathbf{K}$ *is the free negative cone over* $\mathbf{B}$.

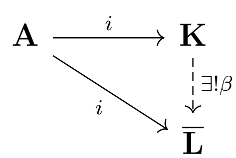
[3]The $\{\backslash, /, \rightarrow, 1\}$-subreducts of Gödel negative cones are easily seen to be axiomatized by the axioms of cone algebras together with the axioms for Hilbert algebras; see, for example, [14].

Proof.

(1) This is a consequence of two facts: (i) the Gödel negative cones form a variety, and therefore the left adjoint of the forgetful functor into any of its reducts always exists; and (ii) the $\{\backslash, /, \to, 1\}$-reduct of a GGMV is a subreduct of a Gödel negative cone, by Corollary 22.

(2) It is a consequence of the following result of [28, Corollary 3.15]: If a cone algebra $\mathbf{C}$ is a subreduct of a negative cone $\mathbf{H}^-$, then the subalgebra of $\mathbf{H}^-$ generated by C is the free extension of $\mathbf{C}$. $\square$

Remark 25. Retaining the notation of the preceding lemma, $\mathbf{A}$ is a subreduct of $\mathbf{K}$. Note that the negative cone $\mathbf{G}^-$ associated with the RL reduct of $\mathbf{M}$ in Theorem 12 is generated as negative cone by $M = B$, and therefore, by [28, Corollary 3.15], $\mathbf{G}^- = \mathbf{L}$. Moreover, by [16, Theorem 3.4.(5)], the RL reduct of $\mathbf{M}$ is contained in $\mathbf{L}$ as a lattice filter — actually, it is the image of a dense nucleus γ on $\mathbf{L}$. Note that $\mathbf{L}$ is a projectable IGMV algebra by Theorem 21; hence, it can be equipped with a Gödel implication $\to$ (Theorem 21), which extends $\to^{\mathbf{A}}$ but is not necessarily the restriction to L of $\to^{\mathbf{K}}$. Therefore, $\mathbf{A}$ is included in the Gödel negative cone $\overline{\mathbf{L}}$ that expands $\mathbf{L}$ by $\to$. Namely, we are in the situation depicted in the figure below. Since $\mathbf{K}$ is the free Gödel negative cone over $\mathbf{A}$, *there exists a unique $\mathcal{GLG}^-$ homomorphism β making the diagram commutative.*



Actually, β is idempotent, whence $\overline{\mathbf{L}}$ is an RL retract of $\mathbf{K}$ in the usual, universal algebraic sense. Therefore, any Gödel GMV algebra $\mathbf{M}$ uniquely determines a triple $(\mathbf{K}, \beta, \gamma)$, where $\mathbf{K}$ is the free Gödel negative cone over $\mathbf{A}$, with β and γ as in the preceding sentences.

We now define the categories we wish to investigate:

- GGMV is the category whose objects are Gödel GMV algebras and whose arrows are their algebra homomorphisms.

- GLG^- is the category whose objects are the triples $(\mathbf{K}, \beta, \gamma)$ such that $\mathbf{K}$ is a Gödel negative cone, β is an idempotent endomorphism on $\mathbf{K}$

and γ is a dense nucleus on its image;[4] and its morphisms are mappings $f : (\mathbf{K_1}, \beta_\mathbf{1}, \gamma_\mathbf{1}) \to (\mathbf{K_2}, \beta_\mathbf{2}, \gamma_\mathbf{2})$ such that f is a $\mathcal{GLG}^-$-homomorphisms that satisfies $f\gamma_\mathbf{1}\beta_\mathbf{1} = \gamma_\mathbf{2} f \beta_\mathbf{1}$, as shown in the next diagram:

$$\begin{array}{ccc}
\mathbf{K}_1 & \xrightarrow{\quad f \quad} & \mathbf{K}_2 \\
{\scriptstyle \beta_1}\big\downarrow & & \big\downarrow{\scriptstyle \beta_2} \\
\overline{\mathbf{L}}_1 & \xrightarrow{f\restriction \overline{\mathbf{L}}_1} & \overline{\mathbf{L}}_2 \\
{\scriptstyle \gamma_1}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_2} \\
\mathbf{M}_1 & \xrightarrow{f\restriction \operatorname{im}\gamma_1} & \mathbf{M}_2
\end{array}$$

It is implicit in the previous definition that $f[L_1] \subseteq L_2$, although there is no assumption in the diagram above that f preserves the Gödel implication in $\overline{\mathbf{L}}_1$, because, as already noted, $\overline{\mathbf{L}}_1$ need not be a subalgebra of $\mathbf{K}_1$. The condition $f\gamma_\mathbf{1}\beta_\mathbf{1} = \gamma_\mathbf{2} f \beta_\mathbf{1}$ expresses the commutativity of the diagram below:

$$\begin{array}{ccc}
\overline{\mathbf{L}}_1 & \xrightarrow{f\restriction \overline{\mathbf{L}}_1} & \overline{\mathbf{L}}_2 \\
{\scriptstyle \gamma_1}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_2} \\
\mathbf{M}_1 & \xrightarrow{f\restriction \operatorname{im}\gamma_1} & \mathbf{M}_2
\end{array}$$

Let $f : \mathbf{M}_1 \to \mathbf{M}_2$ be a homomorphism of Gödel GMV algebras, and let $\mathbf{K}_1$ and $\mathbf{K}_2$ be the free Gödel negative cones over the $\{\backslash, /, \to, 1\}$-reducts $\mathbf{A}_1$ and $\mathbf{A}_2$ of $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Observe that f, as such, restricts to a homomorphism between these reducts. By Lemma 24, $\mathbf{A}_1, \mathbf{A}_2$ respectively embed into the appropriate reducts of the free Gödel negative cones $\mathbf{K}_1$ and $\mathbf{K}_2$:

$$\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{\quad i \quad} & \mathbf{K}_1 \\
{\scriptstyle f}\big\downarrow & \overset{i\circ f}{\searrow} & \big\downarrow{\scriptstyle \exists ! \overline{f}} \\
\mathbf{A}_2 & \xrightarrow{\quad i \quad} & \mathbf{K}_2
\end{array}$$

[4]It should be noted here that in the definition of GLG^- we do not assume that $\mathbf{K}$ is the free Gödel negative cone over a Gödel GMV algebra, and β, γ need not be the special mappings discussed in Remark 25.

Since $\mathbf{K}_1$ is the free Gödel negative cone over $\mathbf{A}_1$, f extends to a unique homomorphism $\overline{f} : \mathbf{K}_1 \to \mathbf{K}_2$. We call $\overline{f}$ the *free extension* of f.

Equipped with this notion, we introduce two assignments $\mathcal{F} : \mathsf{GGMV} \to \mathsf{GLG}^-$ and $\mathcal{G} : \mathsf{GLG}^- \to \mathsf{GGMV}$, with an eye to showing that they are well-defined functors and that they form an adjoint pair between the categories GLG^- and GGMV.

- Given an object $\mathbf{M}$ in GGMV, $\mathcal{F}(\mathbf{M})$ is the triple $(\mathbf{K}, \beta, \gamma)$ determined as in Remark 25, and given a morphism $f : \mathbf{M}_1 \to \mathbf{M}_2$ in GGMV, $\mathcal{F}(f)$ is the free extension $\overline{f}$ of f.
- Given an object $(\mathbf{K}, \beta, \gamma)$ in GLG^-, $\mathcal{G}(\mathbf{K}, \beta, \gamma)$ is the algebra $\gamma[\beta[\mathbf{K}]]$, and given a morphism $f : (\mathbf{K}_1, \beta_1, \gamma_1) \to (\mathbf{K}_2, \beta_2, \gamma_2)$ in GLG^-, $\mathcal{G}(f)$ is $f{\restriction}\mathrm{im}(\gamma_1)$.

Lemma 26. *$\mathcal{F}$ is a functor between the categories GGMV and GLG^-.*

Proof. We already noticed that $\mathcal{F}(\mathbf{M})$ is an object in GLG^-. Next, take any morphism $f : \mathbf{M}_1 \to \mathbf{M}_2$, and let $\mathbf{A}_i, \mathbf{B}_i$ $(i \in \{1, 2\})$ be, respectively, the $(\backslash, /, \to, 1)$-reducts and $(\backslash, /, 1)$-reducts of $\mathbf{M}_i$. Observe that f restricts to a homomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$, which in turn extends to a homomorphism $f^* : \mathbf{L}_1 \to \mathbf{L}_2$, where $\mathbf{L}_i$ $(i \in \{1, 2\})$ is the free negative cone over $\mathbf{B}_i$. We claim that $\overline{f}{\restriction}_{\mathbf{L}_1}$ coincides with f^*. By [16, Theorem 11], $\mathbf{L}_1$ is generated by $\mathbf{B}_1$ as a monoid. Therefore, for any $a \in L_1$, $a = \prod_{i \leq m}^{\mathbf{L}_1} a_i$, with $a_i \in B_1$, for any $i \leq m$. Thus,

$$f^*(a) = f^*\left(\prod_{i \leq m}^{\mathbf{L}_1} a_i\right) = \prod_{i \leq m}^{\mathbf{L}_2} f^*(a_i) = \prod_{i \leq m}^{\mathbf{L}_2} f(a_i) = \prod_{i \leq m}^{\mathbf{K}_2} f(a_i),$$

since $\mathbf{L}_2$ is an RL subalgebra of $\mathbf{K}_2$. Moreover, since $\overline{f}$ extends f,

$$\overline{f}(a) = \overline{f}\left(\prod_{i \leq m}^{\mathbf{K}_1} a_i\right) = \prod_{i \leq m}^{\mathbf{K}_2} \overline{f}(a_i) = \prod_{i \leq m}^{\mathbf{K}_2} f(a_i),$$

whence our claim follows. Now, since β_1 is onto, all we have to show is that the diagram below is commutative.

$$
\begin{array}{ccc}
\mathbf{L}_1 & \xrightarrow{f\restriction \mathbf{L}_1} & \mathbf{L}_2 \\
{\scriptstyle \gamma_1}\big\downarrow & & \big\downarrow{\scriptstyle \gamma_2} \\
\mathbf{M}_1 & \xrightarrow{f\restriction \operatorname{im}\gamma_1} & \mathbf{M}_2
\end{array}
$$

Let $a \in L_1$. There exist $a_1, \ldots, a_m \in A_1$ such that $a = \prod_{i \leq m}^{\mathbf{L}_1} a_i$. So

$$
\begin{aligned}
f\gamma_1(a) &= f\gamma_1\left(\prod_{i\leq m}^{\mathbf{L}_1} a_i\right) = f\left(\prod_{i\leq m}^{\mathbf{M}_1} a_i\right) = \prod_{i\leq m}^{\mathbf{M}_2} f(a_i) \\
&= \gamma_2\left(\prod_{i\leq m}^{\mathbf{L}_2} f(a_i)\right) = \gamma_2\overline{f}\left(\prod_{i\leq m}^{\mathbf{L}_1} a_i\right) = \gamma_2\overline{f}(a).
\end{aligned}
$$

Thus indeed $f\restriction \operatorname{im}\gamma_1 \circ \gamma_1$ equals $\gamma_2 \circ \overline{f}\restriction \mathbf{L}_1$. Finally, it is also easy to check that $\mathcal{F}$ preserves compositions. Therefore $\mathcal{F}$ is a functor between the categories GGMV and GLG^-. □

Lemma 27. *$\mathcal{G}$ is a functor from GLG^- to GGMV.*

Proof. By Theorem 21, $\mathcal{G}(\mathbf{K}, \beta, \gamma)$ is an object in GGMV. Moreover, by the commutativity requirement $f\gamma_{\mathbf{1}}\beta_{\mathbf{1}} = \gamma_{\mathbf{2}}f\beta_{\mathbf{1}}$, $f\restriction \operatorname{im}\gamma$ is a GGMV-morphism and, in particular, it preserves the Gödel implication. □

Theorem 28. *$\mathcal{F}$ and $\mathcal{G}$ are adjoint functors.*

Proof. Let $\mathbf{M}, \widetilde{\mathbf{K}}$ be objects in the categories GGMV and GLG^-, respectively. We want to show that there is a bijective correspondence between $\mathsf{GLG}^-(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}})$ and $\mathsf{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}))$, that is natural in both coordinates. As regards injectivity, let g, h be distinct morphisms in $\mathsf{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}))$. If $\overline{g}$ is the free extension of g, as observed in the proof of Lemma 26, $\mathcal{F}(g)\restriction\mathbf{M} = \overline{g}\restriction\mathbf{M} = g$ and $\mathcal{F}(h) = \overline{h}\restriction\mathbf{M} = h$. Since g, h are assumed to be distinct, $\mathcal{F}(g) \neq \mathcal{F}(h)$. Now, let $g \in \mathsf{GLG}^-(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}})$. Let $\mathcal{F}(\mathbf{M}) = \mathbf{K}$. Notice that, since $\mathbf{K}$ is free over the $\{\backslash, /, \to, 1\}$-reduct of $\mathbf{M}$, both β and γ are uniquely determined up to isomorphism. By the results in [16], there exists a uniquely determined GLG^--homomorphism $\overline{g}$ between the negative cone $\mathbf{L}$ associated with $\mathbf{M}$ and $\widetilde{\mathbf{L}}$ associated with $\widetilde{\mathbf{M}}$ that makes the diagram

$$
\begin{array}{ccc}
\mathbf{K} & \xrightarrow{\overline{g}} & \widetilde{\mathbf{K}} \\
{\scriptstyle \gamma\circ\beta}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\gamma}\circ\widetilde{\beta}} \\
\mathbf{M} & \xrightarrow{g} & \widetilde{\mathbf{M}}
\end{array}
$$

commutative. Arguing as in Lemma 26, it is easy to see that $\overline{g}$ is a morphism from $\mathbf{K}$ to $\widetilde{\mathbf{K}}$, whence $\mathcal{F}$ is onto.

A routine verification shows that the bijection between $\mathsf{GLG}^{-}(\mathcal{F}(\mathbf{M}), \widetilde{\mathbf{K}})$ and $\mathsf{GGMV}(\mathbf{M}, \mathcal{G}(\widetilde{\mathbf{K}}))$ is natural in both $\mathbf{M}$ and $\widetilde{\mathbf{K}}$. Namely, the following diagram commutes for $g \in \mathsf{GGMV}(\mathbf{M}_1, \mathbf{M}_2)$ and $f \in \mathsf{GLG}^{-}(\widetilde{\mathbf{K}}_1, \widetilde{\mathbf{K}}_2)$:

$$\begin{array}{ccc}
\mathsf{GLG}^{-}(\mathcal{F}(\mathbf{M}_2), \widetilde{\mathbf{K}}_1) & \xrightarrow{\Phi_{\mathbf{M}_2, \widetilde{\mathbf{K}}_1}} & \mathsf{GGMV}(\mathbf{M}_2, \mathcal{G}(\widetilde{\mathbf{K}}_1)) \\
{\scriptstyle \mathcal{F}(g)\circ(\,)\circ f}\downarrow & & \downarrow{\scriptstyle g\circ(\,)\circ\mathcal{G}(f)} \\
\mathsf{GLG}^{-}(\mathcal{F}(\mathbf{M}_1), \widetilde{\mathbf{K}}_1) & \xrightarrow{\Phi_{\mathbf{M}_1, \widetilde{\mathbf{K}}_2}} & \mathsf{GGMV}(\mathbf{M}_1, \mathcal{G}(\widetilde{\mathbf{K}}_2))
\end{array}$$

□

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