FREE OBJECTS AND FREE EXTENSIONS IN THE CATEGORY OF FRAMES

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This work is dedicated to Rick Ball on the occasion of his retirement.

Abstract. This article is concerned with free objects and free extensions over posets in the category of frames. Its primary goal is to present novel representations for these objects as subdirect products of certain chains. Constructions for the corresponding objects in the category of bounded distributive lattices are also presented.

1. Discussion and the Main Results

The primary aim of this note is to present novel representations for free objects and free extensions over posets in the category $\mathcal{F}rm$ of frames. Representations for the corresponding objects in the category $\mathcal{D}$ of bounded distributive lattices are also presented.

Free frames are easy to describe, see for example Johnstone [5]. The free frame F over a set X is usually represented as the frame of lower sets of the free meet-semilattice with identity, or more concretely, as the the frame of lower sets of the semilattice of all finite subsets of X ordered by reverse set-inclusion. It can also be shown that F is isomorphic to the ideal completion of the free bounded distributive lattice over X. The latter representation follows from the results of the present paper, but it can also be obtained directly by using the universal properties of free distributive lattices and ideal completions of lattices.

The statements for the main results will require some notation. For any poset P, we write $\bot \oplus P \oplus \top$ for the poset obtained from P by adjoining a new least element $\bot$ and a new greatest element $\top$. Given a nonempty poset $P = (P, \leq)$, let $\mathcal{R}$ denote the set consisting of all total orders on P extending $\leq$. Further, for each $r \in \mathcal{R}$, let P_r denote the chain $\bot \oplus (P, r) \oplus \top$, and let Q_r denote its ideal completion. Finally, let $\alpha : P \to \prod_{r\in\mathcal{R}} P_r \leq \prod_{r\in\mathcal{R}} Q_r$ be the diagonal map $p \mapsto (\dots, p, p, p, \dots)$.

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Theorem 1.1.

(1) *The $\mathcal{D}$-free extension of P is isomorphic to the $\{\bot,\top\}$–sublattice of $\prod_{r\in\mathcal{R}} P_r$ generated by $\alpha(P)$.*
(2) *The free extension of P in $\mathcal{F}rm$ is isomorphic to the subframe of $\prod_{r\in\mathcal{R}} Q_r$ generated by $\alpha(P)$.*

The corresponding representation for the free frame and the free bounded distributive lattice over a nonempty set X is obtained by letting $\mathcal{R}$ be the set of all total orders on X. One can actually be more selective about the orders used in the representations, as the next result shows. Let X be a nonempty set, and let $\mathcal{R} = \{r|r$ is a well-ordered chain on $X\}$. For each $r \in \mathcal{R}$, let $X_r = (X, r)$, and $Y_r = \bot \oplus X_r \oplus \top$. Define $\alpha : X \to \prod_{r\in\mathcal{R}} Y_r$ to be the diagonal map $\alpha(x) = (\ldots, x, x, \ldots)$.

Theorem 1.2.

(1) *The $\mathcal{D}$-free lattice over X is isomorphic to the $\{\bot,\top\}$– sublattice of $\prod_{r\in\mathcal{R}} Y_r$ generated by $\alpha(X)$. Moreover, $\alpha(X)$ is a set of free generators for this lattice.*
(2) *The free frame over X is isomorphic to the subframe of $\prod_{r\in\mathcal{R}} Y_r$ generated by $\alpha(X)$. Moreover, $\alpha(X)$ is a set of free generators for this frame.*

It is well-known that the free distributive lattice over a set of cardinality κ is a subdirect product of chains of cardinality κ (see, for example, Balbes and Dwinger [1], p. 120). The preceding theorem refines this result by making the subdirect product representation concrete and easy to use, and by restricting the class of chains arising in the representation. The results for frames are apparently new.

Corollary 1.2.1. *The free frame extension L of a poset P is a bialgebraic, coherent frame. Moreover, the lattice of compact elements of L is the $\mathcal{D}$-free extension of P.*

Many of the ideas for the aforementioned representations can be traced back to Weinberg [9], Bernau [2], and Powell and Tsinakis [6]. The results in Francello [4] and Powell and Tsinakis [7] suggested the plausibility of such representations for distributive lattices and frames.

A *frame* is a complete lattice L which satisfies the Frame Distributive Law

$$a \wedge \bigvee X = \bigvee \{a \wedge x | x \in X\},$$

for all $a \in L$ and $X \subseteq L$. Such lattices have a least element, which we shall denote by $\bot$, and a greatest element, denoted by $\top$. Frame homomorphisms are functions between frames which preserve arbitrary joins and finite meets (and thus preserve $\bot$ and $\top$). The category of frames and frame homomorphisms will be denoted by $\mathcal{F}rm$. We denote by $\mathcal{D}$ the category consisting of bounded distributive lattices (distributive lattices with $\bot$ and $\top$) and lattice homomorphisms which preserve $\bot$ and $\top$.

Coherent frames will be of particular interest. A *coherent frame* is an algebraic frame whose subposet of compact elements forms a bounded distributive lattice. The coherent frames are isomorphic to ideal completions of bounded distributive lattices.

In what follows, we shall write $S \Subset X$ to indicate that S is a nonempty, finite subset of X. Let $\mathcal{C} = \mathcal{D}$ or $\mathcal{F}rm$. Given a $\mathcal{C}$-object L, we denote the $\mathcal{C}$-object generated by $X \subseteq L$ by $[X]_{\mathcal{C}}$. Also, given an object L in $\mathcal{C}$ which is generated by $X \subseteq L$, L is said to be *$\mathcal{C}$-free over X* if given $M \in Ob\mathcal{C}$ and any function $f : X \to M$, there exists a $\mathcal{C}$-morphism $\overline{f} : L \to M$ such that $\overline{f}_{|X} = f$. It is easy to verify that this $\overline{f}$ is uniquely determined. Finally, if P is a partially ordered set, then $L \in Ob\mathcal{C}$ is the $\mathcal{C}$-free extension of P if P generates L and every order-preserving function from P to a $\mathcal{C}$-object M can be extended to a $\mathcal{C}$ -morphism from L to M.

We will focus primarily on extensions of posets, as it is clear that the $\mathcal{C}$-free object over a nonempty set X can be viewed as the $\mathcal{C}$- free extension of a totally unordered poset of cardinality $|X|$.

2. Proofs of the Results

We begin with a few preliminary lemmas.

Lemma 2.1. *Let T and S be subsets of a poset $P = (P, \leq)$ such that $t \nleq s$ for each $t \in T$ and $s \in S$. Then there exists a total order on P extending $\leq$ such that every element of T exceeds every element of S.*

Proof. We define a new order $\leq'$ on P as follows:
$x \leq' y \iff (x \leq y)$ or $(y \in \bigcup_{t \in T} \uparrow t$ and $x \in \bigcup_{s \in S} \downarrow s)$.
This defines a partial order which preserves $\leq$, and in which every element of T exceeds every element of S. We now extend $\leq'$ to a total order on P as outlined in Crawley and Dilworth [3], page 6. $\square$

The proof of the next lemma can be found in Balbes and Dwinger [1], p. 86.

Lemma 2.2. *Let L and M be bounded distributive lattices. Let X be a nonempty generating subset of L. A function $f : X \to M$ can be extended to a $\mathcal{D}$-morphism $\overline{f} : L \to M$ if and only if whenever T and S satisfy $T \cup S \Subset X$ and $\bigwedge T \leq \bigvee S$, then $\bigwedge f(T) \leq \bigvee f(S)$.*

Lemma 2.3. *Let L and M be frames. Let X be a nonempty generating subset of L such that $\bigwedge T \in K(L)$, for all $T \subseteq_{fin} X$. A function $f : X \to M$ can be extended to a frame morphism $\overline{\overline{f}} : L \to M$ if and only if whenever T and S satisfy $\emptyset \neq T \bigcup S \subseteq X, T \subseteq_{fin} X$, and $\bigwedge T \leq \bigvee S$, then $\bigwedge f(T) \leq \bigvee f(S)$.*

Proof. Assume f can be extended to a frame morphism. Let $\emptyset \neq T \bigcup S \subseteq X, T \subseteq_{fin} X$, and $\bigwedge T \leq \bigvee S$. Then $\bigwedge f(T) = \bigwedge \overline{\overline{f}}(T) = \overline{\overline{f}}(\bigwedge T) \leq \overline{\overline{f}}(\bigvee S) = \bigvee \overline{\overline{f}}(S) = \bigvee f(S)$.

Now assume that the conditions in the statement of the lemma hold. Let $f : X \to M$, where M is a frame. By Lemma 2.2, we can extend f to a $\mathcal{D}$-morphism $\overline{f} : [X]_{\mathcal{D}} \to M$. (Note that $\overline{f}(\bot) = \bot$ and $\overline{f}(\top) = \top$.) Now,

$$L = [X]_{\mathcal{F}rm} = \{\bigvee_{i \in I}(\bigwedge S_i) | S_i \Subset X\} \cup \{\bot, \top\},$$

and

$$[X]_{\mathcal{D}} = \{\bigvee_{1 \leq i \leq n}(\bigwedge S_i) | S_i \Subset X\} \cup \{\bot, \top\}.$$

It is clear that every element x of L can be expressed as a join of elements of $[X]_{\mathcal{D}}$: $x = \bigvee_{i \in I}(\bigwedge S_i), \bigwedge S_i \in [X]_{\mathcal{D}}$. If x has the preceding representation, let $\overline{\overline{f}}(x) = \overline{\overline{f}}(\bigvee_{i \in I}(\bigwedge S_i)) = \bigvee_{i \in I} \overline{f}(\bigwedge S_i)$. We claim that this definition induces a function $\overline{\overline{f}}(x) : L \to M$. Assume $x \in L$ has representations $x = \bigvee_{i \in I}(\bigwedge S_i) = \bigvee_{j \in J}(\bigwedge S_j)$. To show that $\overline{\overline{f}}$ is well-defined, it suffices to show that for each $i \in I, \overline{f}(\bigwedge S_i) \leq \bigvee_{j \in J} \overline{f}(\bigwedge S_j)$. Let $i \in I$. We have $\bigwedge S_i \leq \bigvee_{j \in J}(\bigwedge S_j)$, with $\bigwedge S_i \in K(L)$, by hypothesis. Thus, there exists $J' \subseteq_{fin} J$ such that $\bigwedge S_i \leq \bigvee_{j \in J'}(S_j)$. Therefore, $\overline{f}(\bigwedge S_i) \leq \overline{f}(\bigvee_{j \in J'}(\bigwedge S_j) = \bigvee_{j \in J'} \overline{f}(\bigwedge S_j) \leq \bigvee_{j \in J} \overline{f}(\bigwedge S_j)$. So $\overline{\overline{f}}$ is a well-defined function.

Finally, it is straightforward to verify that $\overline{\overline{f}}$ is a frame morphism that extends f. □

We now present intrinsic characterizations of the $\mathcal{D}$-free extension and the free frame extension of a poset P.

Lemma 2.4. *Let L be a bounded distributive lattice generated by a nonempty subposet P. Then L is the $\mathcal{D}$-free extension of P if and only if whenever T and S satisfy $T \cup S \Subset P$ and $\bigwedge T \leq \bigvee S$, then there exist $t \in T$ and $s \in S$ such that $t \leq s$.*

Proof. In view of Lemma 2.2, the condition is clearly sufficient. Conversely, suppose that L is the $\mathcal{D}$-free extension of P. Let $T \cup S \Subset P$ such that $t \not\leq s$ for all $t \in T$ and $s \in S$. By Lemma 2.1, there exists a total order r on P extending the original partial order such that every element of T exceeds every element of S. Let $P_r = (P, r)$, and let $Q_r = \bot \oplus P_r \oplus \top$. The identity map from P to Q_r extends to a $\mathcal{D}$-morphism from L to Q_r. It follows that $\bigwedge T \not\leq \bigvee S$ in L, since $\bigwedge T > \bigvee S$ in Q_r. □

Corollary 2.0.2. *Let L be a bounded distributive lattice generated by a nonempty subset X. Then L is the $\mathcal{D}$-free lattice over X if and only if whenever T and S satisfy $T \cup S \Subset X$ and $\bigwedge T \leq \bigvee S$, then $T \cap S \neq \emptyset$.*

Lemma 2.5. *Let L be a frame generated by a nonempty subposet P.Then L is the free frame extension of P if and only if whenever T and S satisfy $\emptyset \neq T \cup S \subseteq P$, $T \subseteq_{fin} P$, and $\bigwedge T \leq \bigvee S$, then there exist $t \in T$ and $s \in S$ such that $t \leq s$.*

Proof. Assume L is the free frame extension of P. Let $\emptyset \neq T \cup S \subseteq P$ with T finite and assume that $t \not\leq s$ for all $t \in T$ and $s \in S$. We claim that $\bigwedge T \not\leq \bigvee S$. By Lemma 2.1, there exists a total order r on P which extend the original order and in which every element of T exceeds every element of S. Let $P_r = \bot \oplus (P, r) \oplus \top$ and $Q_r = \mathcal{I}(P_r)$. Since P_r is a chain and thus a distributive lattice, Q_r is a frame. We view Q_r simply as an algebraic lattice with $K(Q_r) = P_r$. By the definition of P_r, it is clear that $\bigwedge T \geq \bigvee S$ in both P_r and Q_r. We claim that, in fact, $\bigwedge T > \bigvee S$; if $\bigwedge T = \bigvee S$, then $t \leq \bigvee S$ for some $t \in T$. However, t is compact in Q_r, which implies that $t \leq s$ for some $s \in S$; this contradicts our original assumption, so $\bigwedge T > \bigvee S$ in Q_r. Now define $f : P \to Q_r$ by $f(p) = p$. This is an order-preserving map, so it can be extended to a frame homomorphism $\overline{f} : L \to Q_r$. Now, if $\bigwedge T \leq \bigvee S$ in L, then $\overline{f}(\bigwedge T) \leq \overline{f}(\bigvee S)$, which implies that $\bigwedge \overline{f}(T) \leq \bigvee \overline{f}(S)$; that is, $\bigwedge T \leq \bigvee S$ in Q_r, which is a contradiction. Therefore, $\bigwedge T \not\leq \bigvee S$.

Conversely, assume the conditions in the statement of the lemma hold. We first claim that $\bigwedge T \in CJP(L)$ for all $T \subseteq_{fin} P$. First, let us assume $T = \emptyset$. We verify that if $\bigwedge \emptyset = \top \leq \bigvee Y$, for some $Y \subseteq L$, then $\top \in Y$. For each $y \in L$, either $y = \bot, y = \top$, or $y = \bigvee_{i \in I}(\bigwedge S_i)$, with $S_i \Subset P$ for each $i \in I$. Note that $Y \neq \{\bot\}$, and if $\bot \in Y$, then $\bigvee Y = \bigvee (Y - \bot)$, so we may assume $\bot \notin Y$. Now let us assume $\top \notin Y$. Then for each $y \in Y$, we can express y as $y = \bigvee_{i \in I}(\bigwedge S_i)$, with $S_i \Subset P$ for each $i \in I$. Thus, $\top = \bigvee Y = \bigvee(\bigvee_{i \in I}(\bigwedge S_i)) \leq \bigvee P$; that is, $\top = \bigwedge \emptyset \leq \bigvee P$, in violation of the hypothesis assumption. Therefore, $\top \in Y$, which implies that $\top \in CJP(L)$.

Now let $T \Subset P$, and assume $\bigwedge T \leq \bigvee Y$, for some $Y \subseteq L$. Again, for each $y \in L$, either $y = \bot, y = \top$, or $y = \bigvee_{i \in I}(\bigwedge S_i)$, with $S_i \Subset P$ for each $i \in I$. Note that if $\top \in Y$, then $\bigwedge T \leq \bigvee Y = \top \in Y$. Arguing as above, we can assume $\bot \notin Y \neq \emptyset$. Now we assume that $\bigwedge T \leq \bigvee_{i \in I}(\bigwedge S_i)$, with $S_i \Subset P$ for each $i \in I$. Our goal is to show that $\bigwedge T \leq \bigwedge S_i$, for some $i \in I$.

We first claim that given such a representation of $\bigwedge T$ and any $f \in \prod_{i \in I} S_i$, there exist $t \in T$ and $i_f \in I$ such that $t \leq f(i_f)$. We know that $\bigwedge T \leq \bigvee_{i \in I}(\bigwedge S_i)$, with $S_i \Subset P$ for each $i \in I$. Thus, $\bigwedge T \leq \bigvee_{i \in I}(\bigwedge S_i) \leq \bigwedge\{\bigvee_{i \in I} f(i) | f \in \prod_{i \in I} S_i\}$. So $\bigwedge T \leq \bigvee_{i \in I} f(i)$ for each $f \in \prod_{i \in I} S_i$. Let $f \in \prod_{i \in I} S_i$. Now, $T \subseteq_{fin} P, \{f(i) | i \in I\} \subseteq P$, and $\bigwedge T \leq \bigvee_{i \in I} f(i)$ imply that, by hypothesis, there exist $t \in T$ and $i_f \in I$ such that $t \leq f(i_f)$.

We finally claim that $\bigwedge T \leq \bigwedge S_i$, for some $i \in I$. Assume not. Then for each $i \in I$, $\bigwedge T \nleq \bigwedge S_i$, so there exists $s_i \in S_i$ such $t \nleq s_i$ for each $t \in T$. Now consider the map $g \in \prod_{i \in I} S_i$ defined by $g(i) = s_i$. Then $t \nleq g(i)$ for all $t \in T$ and $i \in I$. This is a contradiction of the above, so $\bigwedge T \leq \bigwedge S_i$, for some $i \in I$. Thus, $\bigwedge T \in CJP(L)$ for each $T \subseteq_{fin} P$.

Now, let $f : P \to M$ be an order-preserving map. Let $\emptyset \neq T \cup S \subseteq P, T \subseteq_{fin} P$, and $\bigwedge T \leq \bigvee S$. $\bigwedge T \in CJP(L)$ implies that $\bigwedge T \in K(L)$. By hypothesis, then, there exist $t \in T$ and $s \in S$ such that $t \leq s$. So $\bigwedge f(T) \leq f(t) \leq f(s) \leq \bigvee f(S)$. Thus, by Lemma 2.3, f can be extended to a frame homomorphism. □

Corollary 2.0.3. *Let X be a nonempty generating set for a frame L. Then L is the free frame over X if and only if whenever T and S satisfy $\emptyset \neq T \cup S \subseteq X$, $T \subseteq_{fin} X$, and $\bigwedge T \leq \bigvee S$, then $T \cap S \neq \emptyset$.*

We are now ready to establish the main results of this paper.

Proof. (Theorem 1.1)

(1) Let L be the $\{\bot, \top\}$–sublattice of $\prod_{r \in \mathcal{R}} P_r$ generated by $\alpha(P)$. Note that P is order isomorphic to $\alpha(P)$, and hence we need to show that L is the $\mathcal{D}$-free extension of $\alpha(P)$. We shall make use of Lemmas 2.1 and 2.4. Consider nonempty finite subsets $T = \{\alpha(x_1), \ldots, \alpha(x_n)\}$ and $S = \{\alpha(y_1), \ldots, \alpha(y_m)\}$ of $\alpha(P)$ such that $\alpha(x_i) \nleq \alpha(y_j)$, for $i \in \{1, 2, \ldots, n\}$ and all $j \in \{1, 2, \ldots, m\}$. It follows that $x_i \nleq y_j$ in the original order of P and hence, by Lemma 2.1, there exists a total order $r \in \mathcal{R}$ such that every element of the set $\{x_1, \ldots, x_n\}$ exceeds every element of the set $\{y_1, \ldots, y_m\}$. It follows that $\bigvee\{y_1, \ldots, y_m\} < \bigwedge\{x_1, \ldots, x_n\}$ in P_r; hence, $\bigwedge T \nleq \bigvee S$ in L. Also, if $T = \emptyset$, then $\bigvee\{y_1, \ldots, y_m\} < \bigwedge \emptyset = \top$ for any Q_r. Likewise, if $S = \emptyset$, then $\bigvee \emptyset = \bot < \bigwedge\{x_1, \ldots, x_n\}$. In all three cases, $\bigwedge T \nleq \bigvee S$ in L. Thus, by Lemma 2.4, $[\alpha(P)]_{\mathcal{D}}$ is the $\mathcal{D}$-free extension of $\alpha(X)$.

(2) Let L be the subframe of $\prod_{r \in \mathcal{R}} Q_r$ generated by $\alpha(P)$. Assume $T = \{\alpha(x_1), \ldots, \alpha(x_n)\}$, $S = \{\alpha(y_j) | j \in J\}$, and $\alpha(x_i) \nleq \alpha(y_j)$ for each i and j. So for each i and j, $x_i \nleq y_j$ in the original order. By Lemma 2.1, there exists a total order $r \in \mathcal{R}$ such that every element of $\{x_1, \ldots, x_n\}$ exceeds every element of $\{y_j | j \in J\}$. Thus, $\bigvee\{y_j | j \in J\} \leq \bigwedge\{x_1, \ldots, x_n\}$ in Q_r. In fact, $\bigvee\{y_j | j \in J\} < \bigwedge\{x_1, \ldots, x_n\}$; if $\bigvee\{y_j | j \in J\} = \bigwedge\{x_1, \ldots, x_n\}$, then $x_i \leq \bigvee\{y_j | j \in J\}$ for some $1 \leq i \leq n$. But this element is compact in Q_r, so $x_i \leq y_j$ for some $j \in J$. This contradiction implies that $\bigvee\{y_j | j \in J\} < \bigwedge\{x_1, \ldots, x_n\}$; hence, $\bigwedge T \nleq \bigvee S$ in L. The cases $T = \emptyset$ and $S = \emptyset$ can be handled as in the first part of the proof by using the compactness of $\top$ in the chains Q_r. Thus, by Lemma 2.5, $[\alpha(P)]_{\mathcal{F}rm}$ is the free frame extension of $\alpha(P)$. □

Proof. (Theorem 1.2)

This proof follows in the same manner as Theorem 1.1. In the case of a totally unordered set X, each total order will extend the original order. We can then take ideal completions of these total orders; however, it suffices to take all well-ordered chains with added $\perp$ and $\top$ elements. □

Proof. (Corollary 1.2.1)

We have established within the proof of Lemma 2.5 that if L is the free frame extension of P, then $\bigwedge T \in CJP(L)$ for each $T \subseteq_{fin} P$. Thus, it is evident from the representation of L that each element of L is a join of CJP elements of L. Any such frame is known to be bialgebraic. Finally, Theorem 1.1 shows that L is a coherent frame whose lattice of compact elements is the $\mathcal{D}$-free extension of P. □

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