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George Metcalfe[1] · **Constantine Tsinakis**[2]

Abstract In this (part survey) paper, we revisit algebraic and proof-theoretic methods developed by Franco Montagna and his co-authors for proving that the chains (totally ordered members) of certain varieties of semilinear residuated lattices embed into dense chains of these varieties, a key step in establishing standard completeness results for fuzzy logics. Such "densifiable" varieties are precisely the varieties that are generated as quasivarieties by their dense chains. By showing that all dense chains satisfy a certain e-cyclicity equation, we give a short proof that the variety of all semilinear residuated lattices is not densifiable (first proved by Wang and Zhao). We then adapt the Jenei–Montagna standard completeness proof for monoidal t-norm logic to show that any variety of integral semilinear residuated lattices axiomatized by additional lattice-ordered monoid equations is densifiable. We also generalize known results to show that certain varieties of cancellative semilinear residuated lattices are densifiable. Finally, we revisit the Metcalfe–Montagna proof-theoretic approach, which establishes densifiability of a variety via the elimination of a density rule for a suitable hypersequent calculus, focussing on the case of commutative semilinear residuated lattices.

Dedicated to the memory of Franco Montagna.

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1 Introduction

Proving the completeness of an axiom system with respect to an intended semantics is a familiar problem in the study of logical systems. For classical propositional logic, the Lindenbaum-Tarski construction can be used to show that derivability of a formula α is equivalent to the validity of the equation $\alpha \approx 1$ in the class $\mathcal{BA}$ of all Boolean algebras. The goal then is to show that $\alpha \approx 1$ is valid in $\mathcal{BA}$ if and only if it is valid in the standard two-element Boolean algebra **2**. By Birkhoff's theorem for equational classes, this is equivalent to showing that **2** generates $\mathcal{BA}$ as a variety. But also, by the Stone representation theorem, every Boolean algebra embeds into a power of **2**, so this algebra generates $\mathcal{BA}$ even as a quasivariety.

Two of the most important "fuzzy" logics, with intended semantics defined over the real unit interval [0, 1], are Gödel logic G and Łukasiewicz logic Ł. Gödel logic was introduced by Dummett ([1959](#)) as a generalization of finite-valued logics defined by Gödel ([1932](#)). The intended semantics for the logic is provided by the algebra $\mathbf{G} = \langle [0, 1], \wedge, \vee, \rightarrow, 0, 1 \rangle$, while an axiomatization is obtained as an extension of intuitionistic logic with the prelinearity axiom schema $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. Completeness corresponds, as in the classical case, to showing that the variety $\mathcal{GA}$ of Gödel algebras is generated by **G**, which follows from the fact that $\mathcal{GA}$ is generated by countable Gödel chains (totally ordered members of $\mathcal{GA}$) and the observation that any countable Gödel chain embeds into **G**. Proving completeness for Łukasiewicz logic is harder. The axiom system introduced by Łukasiewicz ([1930](#)) was

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shown to be complete for the intended semantics given by the algebra $\langle[0, 1], \rightarrow, \neg\rangle$, where $x \rightarrow y = \max(1, 1 - x + y)$ and $\neg x = 1 - x$, in an unpublished proof by Wajsberg in the 1930s and then a long syntactic proof by Rose and Rosser (1958). A more elegant algebraic proof, introducing MV-algebras and involving the theory of lattice-ordered abelian groups, was provided by Chang the same year (Chang 1958). For further details and references, we refer to Cignoli et al. (2000).
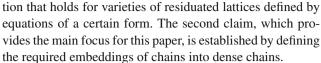
A more general approach to logics with semantics defined over [0, 1] was initiated by Hájek in his 1998 monograph (Hájek 1998). The intended semantics of Hájek's basic fuzzy logic BL is the class of "standard BL-algebras" $\langle[0, 1], \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$, where $\cdot$ is a continuous t-norm: a commutative associative increasing binary function on [0, 1] with unit 1 and residuum $\rightarrow$. Completeness for BL with respect to this intended semantics corresponds to the generation of the variety of BL-algebras by the standard BL-algebras and was proved by Cignoli et al. (essentially by showing that two axioms used by Hájek were redundant) 2 years later (Cignoli et al. 2000). BL thus provides an underlying logic for studying extensions based on particular (classes of) t-norms including Gödel logic, Łukasiewicz logic, and also product logic where the t-norm is just ordinary multiplication (Hájek et al. 1996).

Observing that a t-norm admits a residuum if and only if it is left-continuous, Godo and Esteva introduced monoidal t-norm logic MTL in 2001 (Esteva and Godo 2001) with an intended semantics given by standard MTL-algebras $\langle[0, 1], \wedge, \vee, \cdot, \rightarrow, 0, 1\rangle$, where $\cdot$ is a left-continuous t-norm with residuum $\rightarrow$. Standard completeness for MTL, or, equivalently, generation of the variety of MTL-algebras by the standard MTL-algebras, was proved by Jenei and Montagna a year later (Jenei and Montagna 2002). Their method was subsequently applied to obtain standard completeness results for many other fuzzy logics, including non-commutative, n-contractive, involutive, and first-order versions of MTL (Ciabattoni et al. 2002; Esteva et al. 2002; Montagna and Ono 2002; Jenei and Montagna 2003).

The "Jenei–Montagna" method for a logic L consists of establishing the following:

1. The variety of L-algebras is generated as a quasivariety by its countable chains.
2. Each countable L-chain embeds into a countable dense L-chain.
3. Each countable dense L-chain embeds into a standard L-algebra.

The first claim follows for a broad family of "semilinear" varieties of residuated lattices (see Sect. 2 below), while the third claim is achieved via a Dedekind-MacNeille construc-

tion that holds for varieties of residuated lattices defined by equations of a certain form. The second claim, which provides the main focus for this paper, is established by defining the required embeddings of chains into dense chains.

The introduction of BL, MTL, and a plethora of related logics provided an explicit connection between logics with intended semantics defined over the real unit interval and substructural logics with weakening such as FL_{ew} [see Ono and Komori (1985)]. This connection was witnessed, on the one hand, by the development of hypersequent calculi for many of these logics (Avron 1991; Baaz et al. 2004; Metcalfe et al. 2004, 2005) [see also Metcalfe et al. (2008); Metcalfe (2011)] and, on the other, by the intensive study of their algebraic semantics given by varieties of residuated lattices (Jipsen and Tsinakis 2002; Blount and Tsinakis 2003; Galatos et al. 2007). This led, in the 2003 dissertation of the first author (Metcalfe 2003), to the introduction of new "weakening free" substructural fuzzy logics. In particular, uninorm logic UL was conjectured to be complete with respect to all standard UL-algebras $\langle[0, 1], \wedge, \vee, \cdot, \rightarrow, e, f, 0, 1\rangle$ where $\cdot$ is a left-continuous uninorm: a commutative associative increasing binary function on [0, 1] with unit e and residuum $\rightarrow$. In this case, however, finding the embeddings required by the second step of the Jenei–Montagna method proved to be difficult. Metcalfe and Montagna therefore introduced a new proof-theoretic method in 2007 (Metcalfe and Montagna 2007), proving that the variety of UL-algebras for the logic UL is generated by its dense chains and hence also, via a Dedekind-MacNeille completion, by its standard members. This method was subsequently simplified and extended to other families of logics in Ciabattoni and Metcalfe (2008), Baldi et al. (2012), Baldi (2014) and Baldi and Ciabattoni (2015a, b).

The "Metcalfe–Montagna" method for a logic L consists of the following steps:

1. A hypersequent calculus L is defined for L that is complete with respect to validity in all L-algebras and admits cut elimination.
2. The extension L^D of L with a "density rule" is shown to be complete with respect to validity in all dense L-chains.
3. Density elimination is proved for L^D; that is, a derivation of a hypersequent in L^D can be algorithmically transformed into a derivation of the same hypersequent in L.

Remarkably, in recent papers by Galatos and Horčík (2016) and Baldi and Terui (2016), the method has been reinterpreted algebraically to obtain the embeddings required by the Jenei–Montagna method.

In this (part survey) paper, we revisit the Jenei–Montagna and Metcalfe–Montagna methods for proving that the chains of a variety of semilinear residuated lattices embed into dense

chains of the variety. In Sect. 2, we prove, following related results in the literature [see in particular Metcalfe and Montagna (2007), Ciabattoni and Metcalfe (2008), Metcalfe et al. (2008), Cintula et al. (2009), Cintula and Noguera (2011) and Horčík (2011)], that these "densifiable" varieties are precisely those generated as a quasivariety by their dense chains. By showing that all dense chains satisfy a certain e-cyclicity equation, we then give a short proof of Wang and Zhao's result that the variety of semilinear residuated lattices is not densifiable (Wang and Zhao 2009). In Sect. 3, we adapt the Jenei–Montagna method of Jenei and Montagna (2002) to show that any variety of integral semilinear residuated lattices axiomatized by additional lattice-ordered monoid equations is densifiable. We also generalize methods introduced in Esteva et al. (2002), Horčík (2005) and Horčík et al. (2006) to show that certain varieties of cancellative semilinear residuated lattices are densifiable. In Sect. 4, we describe the Metcalfe–Montagna method of Metcalfe and Montagna (2007), providing a proof of densifiability for the variety of commutative semilinear residuated lattices. We conclude the paper in Sect. 5 with some open problems and directions for further research.

2 Densifiable varieties of semilinear residuated lattices

A *residuated lattice* [see Blount and Tsinakis (2003), Jipsen and Tsinakis (2002), Galatos et al. (2007) and Metcalfe et al. (2010) for further details] is an algebraic structure

$$\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle$$

satisfying the following conditions:

(a) $\langle L, \cdot, e \rangle$ is a monoid;
(b) $\langle L, \wedge, \vee \rangle$ is a lattice with order $\leq$;
(c) $\backslash$ and $/$ are binary operations satisfying the residuation property

$$xy \leq z \iff y \leq x \backslash z \iff x \leq z/y.$$

If $\leq$ is a total order, then we call $\mathbf{L}$ a *(residuated) chain*. If $\leq$ is also dense, then we call $\mathbf{L}$ a *dense chain*. We recall that a residuated lattice is *commutative* if it satisfies $xy \approx yx$, *integral* if it satisfies $x \leq e$, and *idempotent* if it satisfies $xx \approx x$. In commutative residuated lattices, the residuals $x \backslash y$ and y/x coincide and we therefore often replace both with $x \to y$, shortening the signature accordingly. We also define $x^0 = e$ and $x^{n+1} = x \cdot x^n$ for $n \in \mathbb{N}$.

A residuated lattice is called *semilinear* if it is a subdirect product of residuated chains. The class $\mathcal{S}em\mathcal{RL}$ of semilinear residuated lattices forms a variety; a finite equational

basis for $\mathcal{S}em\mathcal{RL}$ is provided in Blount and Tsinakis (2003) [see also Jipsen and Tsinakis (2002) and Botur et al. (2015)]. Any subvariety $\mathcal{V}$ of $\mathcal{S}em\mathcal{RL}$ is clearly generated as a quasivariety by its countable chains; that is, $\mathcal{V} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_U(\mathcal{V}^c)$, where $\mathcal{V}^c$ is the class of countable chains in $\mathcal{V}$ and $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$, and $\mathbb{P}_U$ denote the isomorphism, subalgebra, product, and ultraproduct class operators, respectively. Equivalently, a quasi-equation is valid in $\mathcal{V}$ if and only if it is valid in $\mathcal{V}^c$.

In this paper, we aim to identify varieties of semilinear residuated lattices that are generated as quasivarieties by their dense chains. The following characterization of this property is an easy consequence of Cintula and Noguera (2011) (Theorems 3.4.3 and 3.4.11), by way of Czelakowski and Dziobiak (1990) (Lemma 1.5); for convenience, we provide here a self-contained proof.

Theorem 1 *A variety $\mathcal{V}$ of semilinear residuated lattices is generated as a quasivariety by its dense chains if and only if each chain in $\mathcal{V}$ embeds into a dense chain in $\mathcal{V}$.*

Proof Let $\mathcal{V}$ be a variety of semilinear residuated lattices. For the right-to-left direction, it suffices to recall that every algebra in $\mathcal{V}$ embeds into a product of chains of $\mathcal{V}$. So if each chain in $\mathcal{V}$ embeds into a dense chain in $\mathcal{V}$, then every algebra in $\mathcal{V}$ embeds into a product of dense chains in $\mathcal{V}$.

For the left-to-right direction, suppose that $\mathcal{V}$ is generated as a quasivariety by the class $\mathcal{V}^d$ consisting of the dense chains of $\mathcal{V}$, i.e. $\mathcal{V} = \mathbb{I}\mathbb{S}\mathbb{P}\mathbb{P}_U(\mathcal{V}^d)$. Let $\mathbf{A} \in \mathcal{V}$ be a non-trivial chain. Then, since an ultraproduct of dense chains is again a dense chain, we may assume that $\mathbf{A}$ is a subalgebra of a product $\mathbf{B} = \prod_{i \in I} \mathbf{B}_i$ of dense chains $\mathbf{B}_i$ ($i \in I$). Given $a, b \in B$, let $[a = b] = \{i \in I : a(i) = b(i)\}$.

For each proper filter F on I (that is, a filter of the Boolean algebra $\mathcal{P}(I)$), consider the following congruence relation on $\prod_{i \in I} \mathbf{B}_i$:

$$a \theta_F b \iff [a = b] \in F.$$

Note that $\theta_F \cap A^2$ is a congruence on $\mathbf{A}$ and for filters F, K on I,

$$\theta_F \cap \theta_K = \theta_{F \cap K}.$$

We consider the set of filters

$$\mathcal{F} = \{F \subseteq \mathcal{P}(I) : F \text{ is a proper filter on } I \text{ and } \theta_F \cap A^2 = \Delta_A\}.$$

Observe that $\{I\} \in \mathcal{F} \neq \emptyset$. Moreover, if $\mathcal{C}$ is a chain in $\mathcal{F}$, then $\theta_{\cup \mathcal{C}} \cap A^2 = \Delta_A$, which implies that $\mathcal{C}$ has an upper bound in $\mathcal{F}$. Hence, by Zorn's Lemma, $\mathcal{F}$ has a maximal element U, which is clearly proper. We claim that U is an ultrafilter on I.

Let J be a proper non-empty subset J of I. Let $\uparrow J$ be the principal filter on I generated by J, and $\uparrow(I \setminus J)$ the

principal filter generated by $I \setminus J$. Set $F_1 = U \vee \uparrow J$ and $F_2 = U \vee \uparrow(I \setminus J)$. Then $(\theta_{F_1} \cap A^2) \cap (\theta_{F_2} \cap A^2) = (\theta_{F_1} \cap \theta_{F_2}) \cap A^2 = (\theta_{F_1 \cap F_2}) \cap A^2 = \theta_U \cap A^2 = \Delta_A$. Since $\mathbf{A}$ is a chain, the element Δ_A is finitely meet irreducible in the congruence lattice of $\mathbf{A}$ [see Blount and Tsinakis (2003)]. Hence, either $\theta_{F_1} \cap A^2 = \Delta_A$ or $\theta_{F_2} \cap A^2 = \Delta_A$. But then the maximality of U implies that $J \in U$ or $I \setminus J \in U$, establishing that U is an ultrafilter on I. Since $\theta_U \cap A^2 = \Delta_A$, it follows that $\mathbf{A}$ is isomorphic to the subalgebra $\mathbf{A}/U$ of $\mathbf{B}/U$. We conclude the proof by observing that $\mathbf{B}/U$ is an ultraproduct of dense chains and hence itself a dense chain. □

For convenience, let us call any variety of semilinear residuated lattices satisfying one of the equivalent conditions in Theorem 1 *densifiable*. The next result shows that any densifiable variety of semilinear residuated lattices is e-*cyclic*, that is, it satisfies the e-cyclicity equation $x \backslash e \approx e/x$. This equation plays a key role in the development of Conrad-type theory in the setting of residuated lattices (Botur et al. 2015).

Lemma 1 *Every dense residuated chain satisfies the e-cyclicity equation* $x \backslash e \approx e/x$.[1]

Proof We show that $x \backslash e \approx e/x$ is valid in any dense residuated chain $\mathbf{A}$. Suppose for a contradiction that $a \backslash e > e/a$ for some $a \in A$. Then, using residuation, $a > (a \backslash e) \backslash e$. By assumption, $a > b > (a \backslash e) \backslash e$ for some $b \in A$. So, using residuation again, $e > a \backslash b$ and $(a \backslash e) b > e$. Combining these inequations, $(a \backslash e) b > a \backslash b$, which gives $a(a \backslash e) b > b$. But $e \geq a(a \backslash e)$, so $b = eb \geq a(a \backslash e) b > b$, a contradiction. Hence $a \backslash e \leq e/a$, and reasoning symmetrically, $a \backslash e = e/a$. □

Consider now the three element (idempotent) residuated chain $\mathbf{C}$ with universe $C = \{\bot, e, \top\}$ ordered by $\bot < e < \top$ and multiplication table

$\cdot$	$\bot$	e	$\top$
$\bot$	$\bot$	$\bot$	$\top$
e	$\bot$	e	$\top$
$\top$	$\bot$	$\top$	$\top$

It is easily checked that $\cdot$ is associative and residuated, so $\mathbf{C} = \langle C, \cdot, \backslash, /, \wedge, \vee, e \rangle$ is a residuated chain. But also $\bot \backslash e = e$ and $e/\bot = \top$, so $\mathbf{C}$ is not e-cyclic. We immediately obtain the following result:

Theorem 2 [Wang and Zhao (2009)] *The variety of semilinear residuated lattices is not densifiable.*

The first proof of this theorem, given by Wang and Zhao (2009) (disproving a conjecture in Metcalfe et al. (2008)), followed a similar pattern, but involved a more complicated equation and a much larger algebra. A significantly shorter proof, using an equation with three variables and a four element algebra, was provided by Horčik (2011). It is also noted in Horčík (2011) that the variety of idempotent semilinear residuated lattices is not densifiable, an immediate consequence here of the fact that the algebra $\mathbf{C}$ defined above is idempotent. We should remark that, strictly speaking, the proofs in Horčík (2011) establish the densifiability of varieties of pointed residuated lattices (or FL-algebras): residuated lattices with an extra constant symbol. Since this constant symbol satisfies no extra equations for these varieties, the difference in the results is negligible. However, other failures of densifiability established in Horčík (2011) for involutive varieties of semilinear residuated lattices (again by finding a suitable chain where an equation satisfied by all dense chains fails) are particular to the pointed residuated lattices setting.

3 An algebraic approach

In order to show that a variety of semilinear residuated lattices is densifiable, it suffices to establish a single densification step: embedding chains of the variety containing a gap between two elements into chains of the variety where the gap is filled. More precisely:

Lemma 2 *A variety $\mathcal{V}$ of semilinear residuated lattices is densifiable if and only if any countable chain $\mathbf{A} \in \mathcal{V}$ satisfying $a < b$ for some $a, b \in A$ is a subalgebra of a countable chain $\mathbf{B} \in \mathcal{V}$ satisfying $a < c < b$ for some $c \in B$.*

Proof Let $\mathcal{V}$ be a variety of semilinear residuated lattices. The left-to-right direction is straightforward. Suppose that $\mathcal{V}$ is densifiable and let $\mathbf{A}$ be a countable chain in $\mathcal{V}$ satisfying $a < b$ for some $a, b \in A$. Then, $\mathbf{A}$ is a subalgebra of a dense chain $\mathbf{C}$ in $\mathcal{V}$ and there exists $c \in C$ such that $a < c < b$. We let $\mathbf{B}$ be the countable subalgebra of $\mathbf{C}$ generated by $A \cup \{c\}$. For the right-to-left direction, suppose that any countable chain $\mathbf{A} \in \mathcal{V}$ satisfying $a < b$ for some $a, b \in A$ is a subalgebra of a countable chain $\mathbf{B} \in \mathcal{V}$ satisfying $a < c < b$ for some $c \in B$. To prove that $\mathcal{V}$ is densifiable, it suffices to show that every countable chain $\mathbf{A} \in \mathcal{V}$ embeds into a dense chain $\mathbf{B} \in \mathcal{V}$. Let $\mathbf{A}_0 = \mathbf{A}$. We define for each $n \in \mathbb{N}$, a countable chain $\mathbf{A}_{n+1} \in \mathcal{V}$ such that $\mathbf{A}_n$ is a subalgebra of $\mathbf{A}_{n+1}$ and for all $a, b \in A_n$ satisfying $a < b$, there exists $c \in A_{n+1}$ such that $a < c < b$. Enumerate all pairs $a_i, b_i \in A_n$ satisfying $a_i < b_i$ and define $\mathbf{A}_n^0 = \mathbf{A}_n$ and $\mathbf{A}_n^{i+1}$ for $i \in \mathbb{N}$ as a countable chain in $\mathcal{V}$ (which exists by assumption) such that $\mathbf{A}_n^i$ is a subalgebra of $\mathbf{A}_n^{i+1}$ and for some $c_i \in A_n^{i+1}$, $a_i < c_i < b_i$. Let $\mathbf{A}_{n+1} \in \mathcal{V}$ be the countable limit algebra

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[1] This result was observed independently by Nikolaos Galatos (private communication).

with universe $A_{n+1} = \bigcup_{i \in \mathbb{N}} A_n^i$. Note that for any pair $a < b$ in A_n, there exists $c \in A_{n+1}$ such that $a < c < b$. Finally, let $\mathbf{B} \in \mathcal{V}$ be the countable limit algebra with $B = \bigcup_{n \in \mathbb{N}} A_n$. Clearly, $\mathbf{A}_0$ is a subalgebra of $\mathbf{B}$, which is a dense chain by construction. □

This criterion for densifiability is formulated for commutative semilinear residuated lattices in Metcalfe et al. (2008) and appears also in more general versions in Cintula and Noguera (2011), Baldi and Terui (2016) and Galatos and Horčík (2016).

We will now use this lemma to establish the densifiability of various families of varieties of semilinear residuated lattices defined by equations of a particular form. Let us call a formula built using the operation symbols $\cdot$ and e, a *monoid formula*, and a formula built using $\wedge, \vee, \cdot$, and e, an ℓ-*monoid* (short for *lattice-ordered monoid*) *formula*. We call $\alpha \leq \beta$ a *monoid inequation* if α and β are monoid formulas, and $\alpha \approx \beta$ an ℓ-*monoid equation* if α and β are both ℓ-monoid formulas.

Let us denote the variety of integral semilinear residuated lattices by $\mathcal{S}em\mathcal{IRL}$. The proof of the following theorem generalizes the proof provided by Jenei and Montagna for (bounded) commutative integral semilinear residuated lattices in Jenei and Montagna (2002), extended to other varieties of integral residuated lattices by various authors in Ciabattoni et al. (2002), Esteva et al. (2002), Jenei and Montagna (2003) and Horčík (2011).

Theorem 3 *Any variety defined over $\mathcal{S}em\mathcal{IRL}$ by ℓ-monoid equations is densifiable.*

Proof Let $\mathcal{V}$ be a non-trivial variety defined over $\mathcal{S}em\mathcal{IRL}$ by ℓ-monoid equations. By distributing joins and meets over multiplication and using lattice distributivity, every ℓ-monoid formula is clearly equivalent to either a join of meets of monoid formulas or a meet of joins of monoid formulas. Hence every ℓ-monoid equation can be replaced by inequations with a join of meets of monoid formulas on the left and a meet of joins of monoid formulas on the right. Using standard lattice properties, it follows that $\mathcal{V}$ is axiomatized over $\mathcal{S}em\mathcal{IRL}$ by inequations of the form

$$\alpha_1 \wedge \cdots \wedge \alpha_n \leq \beta_1 \vee \cdots \vee \beta_m$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ are monoid formulas built using the variables $x_1, \ldots, x_k$. We claim that the formulas $\beta_1, \ldots, \beta_m$ can be chosen in such a way that any variable occurring on the right occurs on the left of these inequations. Suppose that this is not the case for some variable x_i. Observe that x_i does not occur in some β_j; otherwise, $e \leq x_i$ holds in all algebras in $\mathcal{V}$, contradicting the fact that $\mathbf{A}$ is a non-trivial algebra. Now let $\alpha_1 \wedge \cdots \wedge \alpha_n \leq \beta_1' \vee \cdots \vee \beta_m'$ be the inequation obtained by substituting x_i with β_j. If the original

inequation holds in an integral semilinear residuated lattice $\mathbf{A}$, then clearly so does the new inequation, as it is a substitution instance of the original inequation. The converse also holds, since, by integrality, $\beta_k' \leq \beta_j$ holds in $\mathbf{A}$ for each k such that x_i occurs in β_k. So we have removed x_i from the equation without changing $\mathcal{V}$. The claim follows by repeating this argument.

Now let $\mathbf{A} \in \mathcal{V}$ be a countable chain and consider $a, b \in A$ satisfying $a < b$. By Lemma 2, it suffices to show that $\mathbf{A}$ is a subalgebra of some countable chain $\mathbf{B} \in \mathcal{V}$ satisfying $a < c < b$ for some $c \in B$. Let us therefore assume that for all $c \in A$, either $c \leq a$ or $b \leq c$. Let $\mathbf{2} = \{0, 1\}$ be the two-element chain and let

$$B = \{(c, 1) : c \in A\} \cup \{(b, 0)\}$$

be the countable subset of the lexicographic product $\mathbf{A} \overrightarrow{\times} \mathbf{2}$ endowed with the restriction of the order of $\mathbf{A} \overrightarrow{\times} \mathbf{2}$. Define

$$(u, r) \cdot^{\mathbf{B}} (v, s) = (uv, 1) \wedge (u, r) \wedge (v, s).$$

Observe first that

$$(u, r)(e, 1) = (ue, 1) \wedge (u, r) \wedge (e, 1) = (u, r)$$
$$= (eu, 1) \wedge (e, 1) \wedge (u, r) = (e, 1)(u, r).$$

Also, if $(u, r) \leq (v, s)$, then $u \leq v$, so $uw \leq vw$ and $wu \leq wv$, and

$$(u, r)(w, t) = (uw, 1) \wedge (u, r) \wedge (w, t)$$
$$\leq (vw, 1) \wedge (v, s) \wedge (w, t) = (v, s)(w, t)$$
$$(w, t)(u, r) = (wu, 1) \wedge (w, t) \wedge (u, r)$$
$$\leq (wv, 1) \wedge (w, t) \wedge (v, s) = (w, t)(v, s).$$

Since $\mathbf{B}$ is a chain, it follows also that

$$((u, r) \wedge (v, s))(w, t) = (u, r)(w, t) \wedge (v, s)(w, t).$$

Hence, for associativity, we obtain

$$((u, r)(v, s))(w, t) = ((uv, 1) \wedge (u, r) \wedge (v, s))(w, t)$$
$$= (uv, 1)(w, t) \wedge (u, r)(w, t)$$
$$\wedge (v, s)(w, t)$$
$$= (uvw, 1) \wedge (uv, 1) \wedge (uw, 1)$$
$$\wedge (vw, 1) \wedge (u, r) \wedge (v, s) \wedge (w, t)$$
$$= (uvw, 1) \wedge (u, r) \wedge (v, s) \wedge (w, t)$$
$$= (u, r)((v, s)(w, t)).$$

For the residuals, we observe that

$$(u, r)(v, s) \leq (w, 1) \iff (uv, 1) \wedge (u, r) \wedge (v, s)$$
$$\leq (w, 1)$$
$$\iff uv \leq w$$
$$\iff v \leq u \backslash w$$
$$\iff (v, s) \leq (u \backslash w, 1),$$

noting that in the second equivalence, if $(uv, 1) \wedge (u, r) \wedge (v, s) \leq (w, 1)$, then either $(uv, 1) \wedge (u, r) \wedge (v, s) = (uv, 1)$, or one of $uv = u \leq w$ and $uv = v \leq w$ holds. Hence, we obtain

$$(u, r) \backslash (w, 1) = (u \backslash w, 1) \quad \text{and, similarly,}$$
$$(w, 1)/(v, s) = (w/v, 1).$$

But also,

$$(u, r)(v, s) \leq (b, 0) \iff (uv, 1) \wedge (u, r) \wedge (v, s) \leq (b, 0)$$
$$\iff (uv, 1) \leq (a, 1) \quad \text{or} \quad (u, r)$$
$$\wedge (v, s) \leq (b, 0),$$

so, writing $\Rightarrow$ for the Heyting implication of $\mathbf{B}$ viewed as a chain, we obtain

$$(u, r) \backslash (b, 0) = (u \backslash a, 1) \vee ((u, r) \Rightarrow (b, 0)) \quad \text{and}$$
$$(b, 0)/(v, s) = (a/v, 1) \vee ((v, s) \Rightarrow (b, 0)).$$

Hence, $\mathbf{B}$ is an integral residuated chain. Now observe that
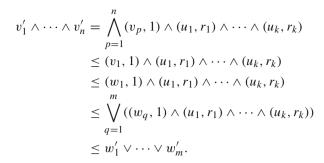
$$(u_1, r_1)(u_2, r_2) \cdots (u_n, r_n) = (u_1 u_2 \ldots u_n, 1)$$
$$\wedge (u_1, r_1) \wedge (u_2, r_2)$$
$$\wedge \cdots \wedge (u_n, r_n).$$

We aim next to show that $\mathbf{B}$ satisfies all defining inequations of $\mathcal{V}$ of the form

$$\alpha_1 \wedge \cdots \wedge \alpha_n \leq \beta_1 \vee \cdots \vee \beta_m$$

where $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ are monoid formulas built using the variables $x_1, \ldots, x_k$ and each x_i occurs on the left. Consider $u_i \in A$ and $r_i \in \{0, 1\}$ for $1 \leq i \leq k$. Let v_p and w_q be the elements of A obtained for $1 \leq p \leq n$ and $1 \leq q \leq m$ by evaluating α_p and β_q with each x_i assigned to u_i. Without loss of generality, we may assume that $v_1 = \min\{v_1, \ldots, v_n\}$ and $w_1 = \max\{w_1, \ldots, w_m\}$. The inequation above holds in $\mathbf{A}$, so $v_1 \leq w_1$. Now let v'_p and w'_q be the elements of B obtained for $1 \leq p \leq n$ and $1 \leq q \leq m$ by evaluating α_p and β_q with each x_i assigned to (u_i, r_i). Then, recalling the definition of the multiplication in $\mathbf{B}$ and the fact that each α_p and β_q are monoid formulas, we obtain

$$v'_1 \wedge \cdots \wedge v'_n = \bigwedge_{p=1}^{n} (v_p, 1) \wedge (u_1, r_1) \wedge \cdots \wedge (u_k, r_k)$$
$$\leq (v_1, 1) \wedge (u_1, r_1) \wedge \cdots \wedge (u_k, r_k)$$
$$\leq (w_1, 1) \wedge (u_1, r_1) \wedge \cdots \wedge (u_k, r_k)$$
$$\leq \bigvee_{q=1}^{m} ((w_q, 1) \wedge (u_1, r_1) \wedge \cdots \wedge (u_k, r_k))$$
$$\leq w'_1 \vee \cdots \vee w'_m.$$

We may therefore conclude that $\mathbf{B} \in \mathcal{V}$ and that $a \mapsto (a, 1)$ is an embedding of $\mathbf{A}$ into $\mathbf{B}$ with $(a, 1) < (b, 0) < (b, 1)$ as required. $\square$

It is worth making a few remarks in connection with the preceding proof. Observe first that if our goal were to embed $\mathbf{A}$ into a totally ordered residuated *semigroup* containing an element between a and b, then we could define multiplication as simply $(u, r) \cdot^{\mathbf{B}} (v, s) = (uv, 1)$ [see Lemma 4.1.2 of Horčík (2011)]. The more complicated version $(u, r) \cdot^{\mathbf{B}} (v, s) = (uv, 1) \wedge (u, r) \wedge (v, s)$ is required here to ensure that $(e, 1)$ is a unit of the multiplication. Observe also that more straightforward options are available for the target algebra $\mathbf{B}$ if we are concerned only with embedding $\mathbf{A}$ into a residuated chain containing an element between a and b. One could, for example, consider the whole algebra $\mathbf{A} \vec{\times} \mathbf{2}$ instead of $\mathbf{B}$ above, defining multiplication in the same manner. More generally, it suffices to construct a totally ordered monoid extension $\mathbf{B}$ of $\mathbf{A}$ that contains c with $a < c < b$. While $\mathbf{B}$ may not possess all residuals, we require that it preserves those in $\mathbf{A}$. Now the embedding of $\mathbf{B}$ into the residuated chain $\mathcal{L}(\mathbf{B})$ of all order-ideals of $\mathbf{B}$ preserves products, residuals, finite joins, and all existing meets. The subalgebra $\mathbf{C}$ of $\mathcal{L}(\mathbf{B})$ generated by B is a countable chain in $\mathcal{SemIRL}$ that satisfies the required density property.

As remarked above, densifiability results have often been stated for varieties in the signature of pointed residuated lattices. The following lemma shows that in an integral setting, adding axioms ensuring that the additional constant $\bot$ denotes the least element makes no difference to the densifiability of the variety.

Lemma 3 *Let $\mathcal{V}$ be a densifiable variety of integral semilinear residuated lattices and let $\mathcal{V}'$ be the variety of integral semilinear bounded residuated lattices defined by the equational theory of $\mathcal{V}$ and $\bot \leq x$. Then $\mathcal{V}'$ is densifiable.*

Proof Let $\mathcal{V}$ and $\mathcal{V}'$ be as in the statement of the lemma and consider any chain $\mathbf{A}' \in \mathcal{V}'$. Then the residuated lattice reduct $\mathbf{A}$ of $\mathbf{A}'$ is in $\mathcal{V}$. By assumption, $\mathbf{A}$ embeds into a dense chain $\mathbf{B} \in \mathcal{V}$. Let $\bot'$ be the image in $\mathbf{B}$ of the bottom element $\bot$ of $\mathbf{A}$ and define C to be the interval $[\bot', e]$. It is clear that C is closed under the lattice and residual operations of $\mathbf{B}$. Also, since $\mathbf{A}$ embeds into $\mathbf{B}$, $\bot'$ is an idempotent element of $\mathbf{B}$ and

$a, b \in C$ implies $\perp' = \perp'\perp' \leq ab \leq$ e, i.e. $ab \in C$. So C with the operations of $\mathbf{B}$ restricted to C forms a subalgebra $\mathbf{C}$ of $\mathbf{B}$. Let $\mathbf{C}'$ be $\mathbf{C}$ with an additional constant interpreted by $\perp'$. Then, $\mathbf{C}'$ is a dense chain in $\mathcal{V}'$, and clearly $\mathbf{A}'$ embeds into $\mathbf{C}'$. $\qquad\square$

A residuated lattice is called *cancellative* if it satisfies $xy/y \approx x$ and $x\backslash xy \approx y$. The variety $\mathcal{CanSemRL}$ of cancellative semilinear residuated lattices then consists of all residuated lattices satisfying the semigroup cancellation laws $xy \approx xz \Rightarrow y \approx z$ and $yx \approx zx \Rightarrow y \approx z$. Important subvarieties of $\mathcal{CanSemRL}$ include the varieties $\mathcal{SemLG}$ and $\mathcal{SemLG}^-$ of semilinear ℓ-groups and negative cones of semilinear ℓ-groups, respectively. The densifiability results below generalize or adapt methods used to prove (standard) completeness results for ΠMTL and related logics in Esteva et al. (2002), Horčík (2005) and Horčík et al. (2006).

Theorem 4 *Any variety defined over $\mathcal{CanSemRL}$ or $\mathcal{SemLG}^-$ by monoid inequations is densifiable.*

Proof Any trivial variety is densifiable, so let $\mathcal{V}$ be a non-trivial variety defined over $\mathcal{CanSemRL}$ or $\mathcal{SemLG}^-$ by monoid inequations. We claim that $\mathcal{V}$ contains a countable non-trivial integral chain $\mathbf{C}$. If $\mathcal{V}$ is a variety of negative cones of lattice-ordered groups, then the free algebra of $\mathcal{V}$ on one generator is in $\mathcal{V}$ and isomorphic to $\langle \mathbb{Z}^-, \wedge, \vee, +, \rightarrow, 0 \rangle$ where $x \rightarrow y = \min(0, y - x)$. If $\mathcal{V}$ is defined over $\mathcal{CanSemRL}$ by monoid inequations, then $\mathcal{V}$ must contain a countable chain with a non-trivial negative cone $\mathbf{C}$. Clearly, $\mathbf{C}$ is semilinear, cancellative, and, since taking the negative cone preserves all inequations between monoid terms, $\mathbf{C}$ is in $\mathcal{V}$.

Now consider any countable chain $\mathbf{A} \in \mathcal{V}$ and $B = A \times C$ with lexicographic order $\leq^{\mathbf{B}}$. The operation

$$(a, x) \cdot^{\mathbf{B}} (b, y) := (ab, xy)$$

is clearly associative with neutral element (e, e) and cancellative. Also, if $(a, x) \leq (b, y)$ then $a < b$ or $a = b$ and $x \leq y$. Hence for each $(c, z) \in B$, $ac \leq bc$ and it follows that either $ac < bc$ or $ac = bc$ and $xz \leq yz$, i.e.

$$(a, x)(c, z) = (ac, xz) \leq (bc, yz) = (b, y)(c, z).$$

Moreover, it is easily checked that

$$(a, x)\backslash^{\mathbf{B}}(b, y) = \begin{cases} (a\backslash b, x\backslash y) & \text{if } a(a\backslash b) = b \\ (a\backslash b, \text{e}) & \text{if } a(a\backslash b) < b \end{cases}$$

$$(a, x)/^{\mathbf{B}}(b, y) = \begin{cases} (a/b, x/y) & \text{if } (a/b)b = a \\ (a/b, \text{e}) & \text{if } (a/b)b < a. \end{cases}$$

Hence, $\mathbf{B}$ is a cancellative residuated chain and a negative cone if $\mathbf{A}$ is also a negative cone. Moreover, $\mathbf{B}$ satisfies any monoid inequation satisfied by $\mathbf{A}$ and $\mathbf{C}$ and is hence in $\mathcal{V}$. Finally, $a \mapsto (a, \text{e})$ is an embedding of $\mathbf{A}$ into $\mathbf{B}$ and whenever $a < b$ in $\mathbf{A}$, also $(a, \text{e}) < (b, c) < (b, \text{e})$ for some $c < \text{e}$ in $\mathbf{C}$. $\qquad\square$

In the context of ℓ-groups, we consider the inverse operation $x^{-1} = x\backslash\text{e} = \text{e}/x$ and call inequations between formulas built using $\cdot$, e, and $^{-1}$ *group inequations*.

Theorem 5 *Any variety defined over $\mathcal{SemLG}$ by group inequations is densifiable.*

Proof Let $\mathcal{V}$ be a non-trivial variety defined over $\mathcal{SemLG}$ by group inequations and consider any countable chain $\mathbf{A} \in \mathcal{V}$. Define $\mathbf{B}$ as the lexicographic product of $\mathbf{A}$ and $\mathbf{Z} = \langle \mathbb{Z}, \wedge, \vee, +, -, 0 \rangle \in \mathcal{V}$, where

$$(a, x) \cdot^{\mathbf{B}} (b, y) = (ab, x + y)$$
$$(a, x)\backslash^{\mathbf{B}}(b, y) = (a\backslash b, y - x)$$
$$(a, x)/^{\mathbf{B}}(b, y) = (a/b, x - y).$$

Then, $\mathbf{B}$ is a totally ordered group. Moreover, all group inequations satisfied by both $\mathbf{A}$ and $\mathbf{Z}$ are satisfied by $\mathbf{B}$, so $\mathbf{B} \in \mathcal{V}$. Observe finally that $a \mapsto (a, 0)$ is an embedding of $\mathbf{A}$ into $\mathbf{B}$ such that whenever $a < b$ in $\mathbf{A}$, also $(a, 0) < (a, 1) < (a, b)$ in $\mathbf{B}$. $\qquad\square$

The previous theorem provides only a limited family of varieties of ℓ-groups closed under lexicographic products by $\mathbb{Z}$. It is shown in Huss (1991) that there are uncountably many varieties of ℓ-groups that are closed with respect to such lexicographic products and an equal number of semilinear varieties that are not. Many of the varieties that do are not of the type described in Theorem 5. Second, lexicographic products can dramatically change the membership of a variety. Indeed, there is an interesting example of a semilinear ℓ-group variety $\mathcal{V}$ such that, if $\mathcal{V}^l$ is the variety generated by all algebras $\{\mathbb{Z} \overset{\rightarrow}{\times} \mathbf{G} \mid \mathbf{G} \in \mathcal{V}\}$, then $[\mathcal{V}, \mathcal{V}^l]$ is an uncountable interval of semilinear varieties.

We turn our attention now to a further characterization of densifiability that is particularly useful in syntactic approaches to this property. Let $\mathbf{Fm}(Y)$ be the formula algebra of the language of residuated lattices over a set of variables Y, writing just $\mathbf{Fm}$ when Y is a fixed countably infinite set of variables X. For any class $\mathcal{K}$ of residuated lattices, we define for $\Sigma \cup \{\alpha\} \subseteq Fm$,

$$\Sigma \vdash_{\mathcal{K}} \alpha \iff \text{for each } \mathbf{A} \in \mathcal{K} \text{ and homomorphism } h\colon \mathbf{Fm} \to \mathbf{A},$$
$$\text{whenever e} \leq h(\beta) \text{ for all } \beta \in \Sigma, \text{ also e} \leq h(\alpha).$$

It is easily shown that $\vdash_{\mathcal{K}}$ is a substitution-invariant consequence relation [see Metcalfe et al. (2010) for details]. Moreover, if $\mathcal{K}$ is a quasivariety, then $\vdash_{\mathcal{K}}$ is finitary: that is, whenever $\Sigma \vdash_{\mathcal{K}} \alpha$, then there exists some finite $\Sigma' \subseteq \Sigma$

such that $\Sigma' \vdash_{\mathcal{K}} \alpha$. If a variety $\mathcal{V}$ is generated as a quasivariety by $\mathcal{K} \subseteq \mathcal{V}$, then quasi-equation is valid in $\mathcal{V}$ if and only if it is valid in $\mathcal{K}$, and for $\Sigma \cup \{\alpha\} \subseteq Fm$,

$$\Sigma \vdash_{\mathcal{V}} \alpha \iff \Sigma \vdash_{\mathcal{K}} \alpha.$$

Hence, any variety $\mathcal{V}$ of semilinear residuated lattices satisfies the following linearity property for $\Sigma \cup \{\alpha, \beta\} \subseteq Fm$:

$$\Sigma \cup \{\alpha \backslash \beta\} \vdash_{\mathcal{V}} \gamma_1 \quad \text{and} \quad \Sigma \cup \{\beta \backslash \alpha\} \vdash_{\mathcal{V}} \gamma_2$$
$$\implies \quad \Sigma \vdash_{\mathcal{V}} \gamma_1 \vee \gamma_2.$$

Moreover, for any variety $\mathcal{V}$ of commutative residuated lattices and $\Sigma \cup \{\alpha, \beta\} \subseteq Fm$, we have the following "local deduction theorem":

$$\Sigma \cup \{\alpha\} \vdash_{\mathcal{V}} \beta \iff \text{there exists } n$$
$$\in \mathbb{N} \text{ such that } \Sigma \vdash_{\mathcal{V}} (\alpha \wedge e)^n \to \beta.$$

We refer to Galatos et al. (2007), Metcalfe et al. (2008) and Cintula and Noguera (2011) for further details and references.

Let $\mathbf{F}_{\mathcal{V}}(Y)$ be the free algebra of a variety $\mathcal{V}$ of residuated lattices on a set of variables $Y \subseteq X$. We denote the image of α under the natural map from $\mathbf{F}m(Y)$ to $\mathbf{F}_{\mathcal{V}}(Y)$ by $\bar{\alpha}$. Given $\Sigma \subseteq Fm(Y)$, we write $\mathbf{F}_{\mathcal{V}}(Y)/\Sigma$ to denote the quotient of $F_{\mathcal{V}}(Y)$ by the convex normal subalgebra of $F_{\mathcal{V}}(Y)$ generated by $\{\bar{\alpha} : \alpha \in \Sigma\}$. We note that for any countable $\mathbf{A} \in \mathcal{V}$ and $Y \subseteq X$ of the same cardinality as $\mathbf{A}$, there exists $\Sigma \subseteq Fm(Y)$ such that $\mathbf{A}$ is isomorphic to $\mathbf{F}_{\mathcal{V}}(Y)/\Sigma$. Note also that for any $\alpha, \beta \in Fm(Y)$,

$$\Sigma \vdash_{\mathcal{V}} \alpha \backslash \beta \iff \mathbf{F}_{\mathcal{V}}(Y)/\Sigma \models \alpha \leq \beta,$$

where $\models$ denotes the usual satisfaction relation of first-order logic. We are now able to prove our further characterization of densifiable varieties of semilinear residuated lattices, established for the commutative case in Metcalfe and Montagna (2007) and generalized to extended languages in Ciabattoni and Metcalfe (2008) and implicational semilinear logics in Cintula and Noguera (2011) and Cintula and Noguera (2016). Let us write $\mathrm{Var}(S_1, \ldots, S_n)$ for the set of variables occurring in some structures (sets, multisets, sequents, or hypersequents) $S_1, \ldots, S_n$ defined over Fm.

Lemma 4 *A variety $\mathcal{V}$ of semilinear residuated lattices is densifiable if and only if for any $\Sigma \cup \{\alpha, \beta, \gamma\} \subseteq Fm$ and variable $x \notin \mathrm{Var}(\Sigma, \alpha, \beta, \gamma)$:*

$$\Sigma \vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \gamma \implies \Sigma \vdash_{\mathcal{V}} (\alpha \backslash \beta) \vee \gamma.$$

Moreover, if $\mathcal{V}$ is a commutative variety, then $\mathcal{V}$ is densifiable if and only if for any $\{\alpha, \beta, \gamma\} \subseteq Fm$

and variable $x \notin \mathrm{Var}(\alpha, \beta, \gamma)$:

$$\vdash_{\mathcal{V}} (\alpha \to x) \vee (x \to \beta) \vee \gamma \implies \vdash_{\mathcal{V}} (\alpha \to \beta) \vee \gamma.$$

Proof Suppose first that $\mathcal{V}$ is densifiable and hence generated as a quasivariety by its dense chains. Consider $\Sigma \cup \{\alpha, \beta, \gamma\} \subseteq Fm$ and $x \notin \mathrm{Var}(\Sigma, \alpha, \beta, \gamma)$. If $\Sigma \not\vdash_{\mathcal{V}} (\alpha \backslash \beta) \vee \gamma$, then for some dense chain $\mathbf{A} \in \mathcal{V}$ and homomorphism $h \colon \mathbf{F}m \to \mathbf{A}$, we have $h(\delta) \geq e$ for each $\delta \in \Sigma$, $h(\alpha) > h(\beta)$, and $h(\gamma) < e$. Because $\mathbf{A}$ is a dense chain, there exists $c \in A$ such that $h(\alpha) > c > h(\beta)$. We define $h' \colon \mathbf{F}m \to \mathbf{A}$ by $h'(y) = c$ if $y = x$, and $h'(y) = h(y)$ otherwise. Then $h(\delta) \geq e$ for each $\delta \in \Sigma$, $h'(\alpha) > h'(x) > h'(\beta)$, and $h'(\gamma) < e$. So $\Sigma \not\vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \gamma$.

For the converse direction, consider $a >_{\mathbf{A}} b$ in a countable residuated chain $\mathbf{A} \in \mathcal{V}$. We need to prove that there exists a residuated chain $\mathbf{B} \in \mathcal{V}$ such that $\mathbf{A}$ is a subalgebra of $\mathbf{B}$ and $a >_{\mathbf{B}} c >_{\mathbf{B}} b$ for some $c \in B$. First, because $\mathbf{A}$ is countable, we can assume that $\mathbf{A} = \mathbf{F}_{\mathcal{V}}(Y)/\Sigma$, for some set of formulas $\Sigma \subseteq Fm(Y)$ such that $x \notin Y$. For $\gamma \in Fm(Y)$, let us write $[\gamma]$ for the equivalence class of $\bar{\gamma}$ in $\mathbf{A}$. Consider $a = [\alpha] >_{\mathbf{A}} [\beta] = b$ and define

$$\Delta = \{\gamma \backslash \delta : \gamma, \delta \in Fm(Y) \text{ and } [\gamma] >_{\mathbf{A}} [\delta]\}$$

Then $\Sigma \not\vdash_{\mathcal{V}} (\alpha \backslash \beta) \vee \bigvee \Delta'$ for any finite $\Delta' \subseteq \Delta$ and hence also, by assumption,

$$(\star) \quad \Sigma \not\vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \bigvee \Delta' \text{ for any finite } \Delta' \subseteq \Delta.$$

Enumerate all pairs of formulas $\langle \gamma_n, \delta_n \rangle$ from $Fm(Y \cup \{x\})$. Let $\Sigma_0 = \Sigma$ and define Σ_{n+1} for $n \in \mathbb{N}$ such that $\Sigma_{n+1} = \Sigma_n \cup \{\gamma_n \backslash \delta_n\}$ or $\Sigma_{n+1} = \Sigma_n \cup \{\delta_n \backslash \gamma_n\}$ and $(\star)$ is satisfied by Σ_{n+1}. If this were not possible at step $n + 1$, then we would have

$$\Sigma_n \cup \{\gamma_n \backslash \delta_n\} \vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \bigvee \Delta_1$$
$$\text{for some finite } \Delta_1 \subseteq \Delta$$
$$\text{and} \quad \Sigma_n \cup \{\delta_n \backslash \gamma_n\} \vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \bigvee \Delta_2$$
$$\text{for some finite } \Delta_2 \subseteq \Delta.$$

But then, by the linearity property,

$$\Sigma_n \vdash_{\mathcal{V}} (\alpha \backslash x) \vee (x \backslash \beta) \vee \bigvee (\Delta_1 \cup \Delta_2),$$

a contradiction. Let $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma_n$ and define $\mathbf{B} = \mathbf{F}_{\mathcal{V}}(Y \cup \{x\})/\Sigma^*$. Then, $\mathbf{B}$ is a chain by construction. Also, writing now $[\gamma]$ for the equivalence class of $\bar{\gamma}$ in $\mathbf{B}$, we have $[\alpha] >_{\mathbf{B}} [x] >_{\mathbf{B}} [\beta]$. Finally, $\mathbf{A}$ can be viewed as a subalgebra of $\mathbf{B}$ by construction: if $[\gamma] >_{\mathbf{A}} [\delta]$, then $\Sigma \not\vdash_{\mathcal{V}} \gamma \backslash \delta$ and so also $\Sigma^* \not\vdash_{\mathcal{V}} \gamma \backslash \delta$, and $[\gamma] >_{\mathbf{B}} [\delta]$.

Suppose now that $\mathcal{V}$ is a commutative variety. The right-to-left direction follows immediately from the more general

case above. For the other direction, suppose that for any $\{\alpha, \beta, \gamma\} \subseteq Fm$ and $x \notin \mathrm{Var}(\alpha, \beta, \gamma)$, whenever $\vdash_{\mathcal{V}} (\alpha \to x) \vee (x \to \beta) \vee \gamma$, also $\vdash_{\mathcal{V}} (\alpha \to \beta) \vee \gamma$. We prove that this implication holds also in the presence of a set of formulas Σ and hence that $\mathcal{V}$ is densifiable. Consider $\Sigma \cup \{\alpha, \beta, \gamma\} \subseteq Fm$ and $x \notin \mathrm{Var}(\Sigma, \alpha, \beta, \gamma)$ such that

$$\Sigma \vdash_{\mathcal{V}} (\alpha \to x) \vee (x \to \beta) \vee \gamma.$$

By the local deduction theorem, for some $\{\delta_1, \ldots, \delta_m\} \subseteq \Sigma$ and $\delta = (\delta_1 \wedge \mathrm{e}) \cdots (\delta_m \wedge \mathrm{e})$,

$$\vdash_{\mathcal{V}} \delta \to ((\alpha \to x) \vee (x \to \beta) \vee \gamma)$$

and, using some valid equations of commutative semilinear residuated lattices,

$$\vdash_{\mathcal{V}} ((\delta \cdot \alpha) \to x) \vee (x \to (\delta \to \beta)) \vee (\delta \to \gamma).$$

So then, by assumption,

$$\vdash_{\mathcal{V}} ((\delta \cdot \alpha) \to (\delta \to \beta)) \vee (\delta \to \gamma),$$

and, using some valid equations of commutative semilinear residuated lattices,

$$\vdash_{\mathcal{V}} (\delta \cdot \delta) \to ((\alpha \to \beta) \vee \gamma).$$

But $\delta \cdot \delta$ is of the form $(\delta_1' \wedge \mathrm{e}) \cdots (\delta_k' \wedge \mathrm{e})$ for some $\{\delta_1', \ldots, \delta_k'\} \subseteq \Sigma$, so by the local deduction theorem once more, $\Sigma \vdash_{\mathcal{V}} (\alpha \to \beta) \vee \gamma$. $\qquad\square$

Let us remark that the proof of the second part of this theorem for commutative varieties of semilinear residuated lattices makes essential use of the local deduction theorem for such varieties. We do not know, however, whether or not the statement holds in the more general setting of semilinear residuated lattices.

4 A proof-theoretic approach

In this section, we describe a proof-theoretic method for establishing the densifiability of semilinear varieties, introduced by Metcalfe and Montagna in Metcalfe and Montagna (2007) and developed further in Ciabattoni and Metcalfe (2008), Baldi et al. (2012), Baldi (2014) and Baldi and Ciabattoni (2015a, b). For convenience and clarity of exposition, we focus here on just one fundamental example: the variety $\mathcal{CSemRL}$ of commutative semilinear residuated lattices. By Lemma 2, we know that in any densifiable variety, a countable chain containing a gap can be embedded into a countable chain where the gap has been filled by at least one element.

However, it can be a challenging problem—for non-integral varieties in particular—to find these embeddings. Indeed, recent work of Galatos and Horčik (2016) and Baldi and Terui (2016) demonstrates the usefulness of proceeding in the opposite direction: appropriate embeddings for $\mathcal{CSemRL}$ and other varieties have been obtained via an analysis of the corresponding proof-theoretic approach.

Recall that by Lemma 4, it suffices for the densifiability of $\mathcal{CSemRL}$ to show that for any $\{\alpha, \beta, \gamma\} \subseteq Fm$ and $x \notin \mathrm{Var}(\alpha, \beta, \gamma)$, whenever $\vdash_{\mathcal{CSemRL}} (\alpha \to x) \vee (x \to \beta)$, also $\vdash_{\mathcal{CSemRL}} (\alpha \to \beta) \vee \gamma$. We could try to establish this property by considering derivations of corresponding equations in equational logic or formulas in a suitable axiom system. However, in such proof systems, we have very little control over the formulas that occur in derivations. Instead, we make use here of a hypersequent calculus that is not only sound and complete with respect to validity in $\mathcal{CSemRL}$, but also admits cut elimination, allowing us to consider only derivations built from subformulas of the formula to be proved.

Let us define a *sequent* as an ordered pair consisting of a multiset of formulas Γ and a formula α, written $\Gamma \Rightarrow \alpha$. A *hypersequent* is a finite multiset of sequents, written

$$\Gamma_1 \Rightarrow \alpha_1 \mid \ldots \mid \Gamma_n \Rightarrow \alpha_n.$$

Hypersequent rules are sets of rule instances, each consisting of a finite set of hypersequents called the premises of the rule and a further hypersequent called the conclusion. These are typically presented schematically using $\alpha, \beta, \gamma, \delta$ as metavariables for formulas, $\Gamma, \Pi, \Delta, \Sigma$ as metavariables for finite multisets of formulas, and $\mathcal{G}, \mathcal{H}$ as metavariables for hypersequents. A hypersequent calculus is just a set of hypersequent rules. In Fig. 1, we present a hypersequent calculus CSemRL for the variety of commutative semilinear residuated lattices; we also define CSemRL° to be CSemRL without (CUT).

A derivation of a non-empty hypersequent $\mathcal{G}$ in a calculus S is a finite labelled tree such that the root node is labelled $\mathcal{G}$ and for each node labelled $\mathcal{G}_0$ with child nodes labelled $\mathcal{G}_1, \ldots, \mathcal{G}_n$, there is a rule instance of a rule of S with premises $\mathcal{G}_1, \ldots, \mathcal{G}_n$ and conclusion $\mathcal{G}_0$. Note in particular that a derivation in CSemRL will have leaves labelled with hypersequents of the form $\mathcal{G} \mid \alpha \Rightarrow \alpha$, corresponding to the rule (ID) whose instances have no premises. We write $d \vdash_{\mathsf{S}} \mathcal{G}$ to denote that there is a derivation d of $\mathcal{G}$ in S, or just $\vdash_{\mathsf{S}} \mathcal{G}$ if the particular derivation is unimportant. Note that $\vdash_{\mathsf{S}}$ can also be defined as a consequence relation between hypersequents, where hypersequents on the left can label leaves in a derivation, but we will not need this here [see Metcalfe et al. (2010) for further details].
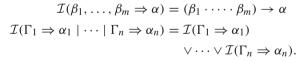
Fig. 1 The hypersequent calculus **CSemRL**

Axioms

$$\frac{}{\mathcal{G} \mid \alpha \Rightarrow \alpha} \ (\text{ID})$$

Cut Rule

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \alpha \quad \mathcal{G} \mid \Pi, \alpha \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \delta} \ (\text{CUT})$$

Structural Rules

$$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \ (\text{EW}) \qquad \frac{\mathcal{G} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}} \ (\text{EC}) \qquad \frac{\mathcal{G} \mid \Gamma, \Sigma \Rightarrow \gamma \quad \mathcal{G} \mid \Pi, \Delta \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Delta \Rightarrow \gamma \mid \Pi, \Sigma \Rightarrow \delta} \ (\text{COM})$$

Left Operation Rules

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \text{e} \Rightarrow \delta} \ (\text{e}\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Pi \Rightarrow \alpha \quad \mathcal{G} \mid \Gamma, \beta \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Pi, \alpha \rightarrow \beta \Rightarrow \delta} \ (\rightarrow\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma, \alpha, \beta \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \alpha \cdot \beta \Rightarrow \delta} \ (\cdot\Rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma, \alpha \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \alpha \wedge \beta \Rightarrow \delta} \ (\wedge\Rightarrow)_1$$

$$\frac{\mathcal{G} \mid \Gamma, \beta \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \alpha \wedge \beta \Rightarrow \delta} \ (\wedge\Rightarrow)_2$$

$$\frac{\mathcal{G} \mid \Gamma, \alpha \Rightarrow \delta \quad \mathcal{G} \mid \Gamma, \beta \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \alpha \vee \beta \Rightarrow \delta} \ (\vee\Rightarrow)$$

Right Operation Rules

$$\frac{}{\mathcal{G} \mid \Rightarrow \text{e}} \ (\Rightarrow\text{e})$$

$$\frac{\mathcal{G} \mid \Gamma, \alpha \Rightarrow \beta}{\mathcal{G} \mid \Gamma \Rightarrow \alpha \rightarrow \beta} \ (\Rightarrow\rightarrow)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \alpha \quad \mathcal{G} \mid \Pi \Rightarrow \beta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \alpha \cdot \beta} \ (\Rightarrow\cdot)$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \alpha}{\mathcal{G} \mid \Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow\vee)_1$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \beta}{\mathcal{G} \mid \Gamma \Rightarrow \alpha \vee \beta} \ (\Rightarrow\vee)_2$$

$$\frac{\mathcal{G} \mid \Gamma \Rightarrow \alpha \quad \mathcal{G} \mid \Gamma \Rightarrow \beta}{\mathcal{G} \mid \Gamma \Rightarrow \alpha \wedge \beta} \ (\Rightarrow\wedge)$$

Example 1 We derive the prelinearity law in **CSemRL** as follows:

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{x \Rightarrow x} \ (\text{ID}) \quad \overline{y \Rightarrow y} \ (\text{ID})}{x \Rightarrow y \mid y \Rightarrow x} \ (\text{COM})}{x \Rightarrow y \mid \Rightarrow y \rightarrow x} \ (\Rightarrow\rightarrow)}{\Rightarrow x \rightarrow y \mid \Rightarrow y \rightarrow x} \ (\Rightarrow\rightarrow)}{\Rightarrow x \rightarrow y \mid \Rightarrow (x \rightarrow y) \vee (y \rightarrow x)} \ (\Rightarrow\vee)_2}{\Rightarrow (x \rightarrow y) \vee (y \rightarrow x) \mid \Rightarrow (x \rightarrow y) \vee (y \rightarrow x)} \ (\Rightarrow\vee)_1}{\Rightarrow (x \rightarrow y) \vee (y \rightarrow x)} \ (\text{EC})$$

Notice that the hypersequent $(x \Rightarrow y \mid y \Rightarrow x)$ two lines down can be read as just a "hypersequent translation" of $(x \rightarrow y) \vee (y \rightarrow x)$.

We interpret sequents and non-empty hypersequents by the function

$$\mathcal{I}(\beta_1, \ldots, \beta_m \Rightarrow \alpha) = (\beta_1 \cdot \cdots \cdot \beta_m) \rightarrow \alpha$$
$$\mathcal{I}(\Gamma_1 \Rightarrow \alpha_1 \mid \cdots \mid \Gamma_n \Rightarrow \alpha_n) = \mathcal{I}(\Gamma_1 \Rightarrow \alpha_1)$$
$$\vee \cdots \vee \mathcal{I}(\Gamma_n \Rightarrow \alpha_n).$$

The following soundness, completeness, and cut elimination results are proved in Metcalfe and Montagna (2007).

Theorem 6 *For any non-empty hypersequent $\mathcal{G}$:*

$$\vdash_{\mathcal{CSemRL}} \mathcal{I}(\mathcal{G}) \iff \vdash_{\text{CSemRL}} \mathcal{G} \iff \vdash_{\text{CSemRL}^\circ} \mathcal{G}.$$

General approaches to defining sequent and hypersequent calculi for varieties of (semilinear) residuated lattices and establishing cut elimination are described in Ciabattoni and Metcalfe (2008), Metcalfe et al. (2008), Ciabattoni et al. (2008) and Ciabattoni et al. (2012, 2016). Let us just remark here that calculi for integral commutative semilinear residuated lattices and idempotent integral commutative semilinear residuated lattices are obtained by extending **CSemRL** with,

respectively, a weakening rule (WL), and both (WL) and a contraction rule (CL):

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \delta} \ \text{(WL)} \qquad \dfrac{\mathcal{G} \mid \Gamma, \Pi, \Pi \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \delta} \ \text{(CL)}$$

By adding also axioms for a constant $\bot$ of the form $\Gamma, \bot \Rightarrow \delta$, we obtain calculi for the varieties of MTL-algebras and Gödel algebras. However, to obtain calculi for varieties of (bounded) pointed residuated lattices, we should, in general, adapt the definition of a sequent slightly to allow an empty right-hand side.

We now consider an appropriate "density" rule that corresponds to the condition in Lemma 4.

$$\dfrac{\mathcal{G} \mid \Gamma \Rightarrow x \mid \Pi, x \Rightarrow \delta}{\mathcal{G} \mid \Gamma, \Pi \Rightarrow \delta} \ \text{(DENSITY)} \quad x \notin \mathrm{Var}(\mathcal{G}, \Gamma, \Pi, \delta).$$

A version of the density rule was introduced by Takeuti and Titani in the context of first-order Gödel logic in Takeuti and Titani (1984), and a first constructive density elimination procedure was given for a hypersequent calculus for this logic by Baaz and Zach (2000).

We define $\mathsf{CSemRL}^{\mathsf{D}}$ to be the calculus CSemRL extended with (DENSITY). We will show below that this extended calculus admits *density elimination*: that is, any derivation in $\mathsf{CSemRL}^{\mathsf{D}}$ can be transformed into a derivation in CSemRL. The transformation proceeds [following Ciabattoni and Metcalfe (2008)] by removing applications of the rule that are uppermost in a derivation. For example, suppose that we have a derivation d ending

$$\dfrac{\genfrac{}{}{0pt}{}{\vdots}{\Gamma \Rightarrow x \mid \Pi, x \Rightarrow \delta}}{\Gamma, \Pi \Rightarrow \delta} \ \text{(DENSITY)}$$

Intuitively, we would like to replace occurrences of x in d "asymmetrically": with Γ if x occurs on the left, and with Π on the left and δ on the right, if x occurs on the right. What we obtain might no longer be a derivation, but is still a finite tree labelled with hypersequents, now ending

$$\dfrac{\genfrac{}{}{0pt}{}{\vdots}{\Gamma, \Pi \Rightarrow \delta \mid \Pi, \Gamma \Rightarrow \delta}}{\Gamma, \Pi \Rightarrow \delta}$$

The last step is an application of (EC) and applications of the operational rules, and most structural rules are preserved by this replacement. Where the derivation potentially breaks down is in applications of (COM) where xs can occur in premises on both the left and the right. For instance, if d ends with

$$\dfrac{\overline{x \Rightarrow x} \ \text{(ID)} \quad \dfrac{\vdots}{\Gamma', \Pi \Rightarrow \delta}}{\Gamma' \Rightarrow x \mid \Pi, x \Rightarrow \delta} \ \text{(COM)}$$

$$\dfrac{\dfrac{\vdots}{\Gamma \Rightarrow x \mid \Pi, x \Rightarrow \delta}}{\Gamma, \Pi \Rightarrow \delta} \ \text{(DENSITY)}$$

then replacing xs as suggested, we get

$$\dfrac{\Gamma, \Pi \Rightarrow \delta \quad \dfrac{\vdots}{\Gamma', \Pi \Rightarrow \delta}}{\Gamma', \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta} \ \text{(COM)}$$

$$\dfrac{\dfrac{\vdots}{\Gamma, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta}}{\Gamma, \Pi \Rightarrow \delta} \ \text{(EC)}$$

But now we are missing the sub-derivation of $(\Gamma, \Pi \Rightarrow \delta)$, which was what we wanted to prove in the first place. However, in this case, we can simply replace the application of (COM) with an application of (EW) and remove the occurrence of $(\Gamma, \Pi \Rightarrow \delta)$ as a premise. Indeed, we can in general use applications of (CUT) to repair such derivations.

Theorem 7 $\mathsf{CSemRL}^{\mathsf{D}}$ *admits density elimination.*

Proof We first introduce some useful notation. Let us assume that (subscripted) λ and μ denote non-negative integers, and, for any multiset of formulas Γ, let Γ^{λ} denote the multiset union of λ copies of Γ. We use $[,]$ to denote multisets and the symbol $\uplus$ for multiset union. Given hypersequents

$$\mathcal{G} = ([\Gamma_i \Rightarrow x]_{i=1}^{n} \mid [\Pi_j, [x]^{\lambda_j}$$
$$\Rightarrow \gamma_j]_{j=1}^{m} \mid [\Sigma_k, [x]^{\mu_k+1} \Rightarrow x]_{k=1}^{l})$$
$$\mathcal{H}_x = (\mathcal{H} \mid \Gamma \Rightarrow x \mid \Pi, x \Rightarrow \delta)$$

where $x \notin \mathrm{Var}(\Gamma_1, \ldots, \Gamma_n, \Pi_1, \ldots, \Pi_m, \gamma_1, \ldots, \gamma_m, \Sigma_1, \ldots, \Sigma_l, \mathcal{H}, \Gamma, \Pi, \delta)$, we define

$$(\mathcal{G}, \mathcal{H}_x)^{\mathcal{D}} = (\mathcal{H} \mid [\Gamma_i, \Pi \Rightarrow \delta]_{i=1}^{n} \mid [\Pi_j, \Gamma^{\lambda_j} \Rightarrow \gamma_j]_{j=1}^{m}$$
$$\mid [\Sigma_k, \Gamma^{\mu_k} \Rightarrow \mathrm{e}]_{k=1}^{l}).$$

Then, it is sufficient to establish the following.

Claim *If* $d_1 \vdash_{\mathsf{CSemRL}^{\circ}} \mathcal{G}$ *and* $d_2 \vdash_{\mathsf{CSemRL}^{\circ}} \mathcal{H}_x$, *then* $\vdash_{\mathsf{CSemRL}} (\mathcal{G}, \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma, \Pi \Rightarrow \delta$.

To see that this suffices observe that an uppermost application of (DENSITY) can be eliminated. Let $\mathcal{G} = (\mathcal{G}' \mid \Gamma \Rightarrow x \mid \Pi, x \Rightarrow \delta)$ be the premise of such an application and suppose that $\vdash_{\mathsf{CSemRL}} \mathcal{G}$. Then by cut elimination, $\vdash_{\mathsf{CSemRL}^{\circ}} \mathcal{G}$ and it follows from the claim applied with $\mathcal{H}_x = \mathcal{G}$ that $\vdash_{\mathsf{CSemRL}} \mathcal{G}' \mid \mathcal{G}' \mid \Gamma, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta$. So by (EC), we obtain $\vdash_{\mathsf{CSemRL}} \mathcal{G}' \mid \Gamma, \Pi \Rightarrow \delta$ as required.

We prove the claim by induction on the height of d_1. If $\mathcal{G} = (\mathcal{G}' \mid x \Rightarrow x)$ or $\mathcal{G} = (\mathcal{G}' \mid \alpha \Rightarrow \alpha)$ for some other formula α, then the result follows by $(\Rightarrow e)$ or (ID), respectively. Otherwise, we consider the last rule applied in d_1. The cases of (EC) and (EW) are immediate using the induction hypothesis. For the operational rules, we have many cases that follow a common pattern. Suppose for example that d_1 ends with

$$\frac{\vdots \qquad\qquad \vdots}{\mathcal{G}' \mid \Sigma_{1,1}, [x]^{\mu'} \Rightarrow \alpha \quad \mathcal{G}' \mid \Sigma_{1,2}, \beta, [x]^{\mu_1+1-\mu'} \Rightarrow x} \\ \frac{}{\mathcal{G}' \mid \Sigma_{1,1}, \Sigma_{1,2}, \alpha \to \beta, [x]^{\mu_1+1} \Rightarrow x} \, (\to\Rightarrow)$$

where $\Sigma_1 = \Sigma_{1,1} \uplus \Sigma_{1,2} \uplus [\alpha \to \beta]$ and $\mathcal{G}' = ([\Gamma_i \Rightarrow x]_{i=1}^n \mid [\Pi_j, [x]^{\lambda_j} \Rightarrow \gamma_j]_{j=1}^m \mid [\Sigma_k, [x]^{\mu_k+1} \Rightarrow x]_{k=2}^l)$. There are two subcases:

(1) If $\mu' < \mu_1 + 1$, then using the induction hypothesis twice:

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_{1,1}, \Gamma^{\mu'} \Rightarrow \alpha \mid \Gamma, \Pi \Rightarrow \delta$$
$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_{1,2}, \beta, \Gamma^{\mu_1-\mu'} \Rightarrow e \mid \Gamma, \Pi \Rightarrow \delta.$$

So by an application of $(\to\Rightarrow)$,

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_1, \Gamma^{\mu_1} \Rightarrow e \mid \Gamma, \Pi \Rightarrow \delta.$$

(2) If $\mu' = \mu_1 + 1$, then using the induction hypothesis twice:

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_{1,1}, \Gamma^{\mu_1+1} \Rightarrow \alpha \mid \Gamma, \Pi \Rightarrow \delta.$$
$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_{1,2}, \beta, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta.$$

So by an application of $(\to\Rightarrow)$:

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_1, \Gamma^{\mu_1+1}, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta.$$

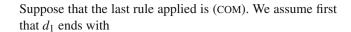But clearly also, using $(\Rightarrow e)$,

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Rightarrow e \mid \Gamma, \Pi \Rightarrow \delta.$$

Hence by an application of (COM),

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Sigma_1, \Gamma^{\mu_1} \Rightarrow e \mid \Gamma, \Pi \Rightarrow \delta \mid \Gamma, \Pi \Rightarrow \delta,$$

and the desired result follows by a further application of (EC).

Suppose that the last rule applied is (COM). We assume first that d_1 ends with

$$\frac{\vdots \qquad\qquad\qquad \vdots}{\mathcal{G}' \mid \Gamma_{1,1}, \Pi_{1,1}, [x]^{\lambda'+1} \Rightarrow x \quad \mathcal{G}' \mid \Gamma_{1,2}, \Pi_{1,2}, [x]^{\lambda_1-\lambda'-1} \Rightarrow \gamma_1} \\ \frac{}{\mathcal{G}' \mid \Gamma_1 \Rightarrow x \mid \Pi_1, [x]^{\lambda_1} \Rightarrow \gamma_1} \, (\text{COM})$$

where $\Gamma_1 = \Gamma_{1,1} \uplus \Gamma_{1,2}$, $\Pi_1 = \Pi_{1,1} \uplus \Pi_{1,2}$, and

$$\mathcal{G}' = ([\Gamma_i \Rightarrow x]_{i=2}^n \mid [\Pi_j, [x]^{\lambda_j} \Rightarrow \gamma_j]_{j=2}^m \mid [\Sigma_k, [x]^{\mu_k+1} \Rightarrow x]_{k=1}^l).$$

Our goal is to show that

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma_1, \Pi \Rightarrow \delta \mid \Pi_1, \Gamma^{\lambda_1} \Rightarrow \gamma_1.$$

By the induction hypothesis,

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma_{1,2}, \Pi_{1,2}, \Gamma^{\lambda_1-\lambda'-1} \Rightarrow \gamma_1,$$

and, using an application of $(e\Rightarrow)$,

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma_{1,2}, \Pi_{1,2}, \Gamma^{\lambda_1-\lambda'-1}, e \Rightarrow \gamma_1.$$

But also by the induction hypothesis,

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma_{1,1}, \Pi_{1,1}, \Gamma^{\lambda'} \Rightarrow e,$$

and an application of (CUT) yields

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma_1, \Pi_1, \Gamma^{\lambda_1-1} \Rightarrow \gamma_1.$$

Now let $\cdot(\Gamma_1)$ be the product of the formulas in Γ_1. Using applications of (EW), $(e\Rightarrow)$, and $(\cdot\Rightarrow)$, we obtain

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \cdot(\Gamma_1), \Pi_1, \Gamma^{\lambda_1-1} \Rightarrow \gamma_1 \mid \Pi, \cdot(\Gamma_1) \Rightarrow \delta.$$

Moreover, by substituting all occurrences of x with $\cdot(\Gamma_1)$ in the derivation d_2 and adding an application of (EW), we obtain

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma \Rightarrow \cdot(\Gamma_1) \mid \Pi, \cdot(\Gamma_1) \Rightarrow \delta.$$

Hence, an application of (CUT) yields

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Pi_1, \Gamma^{\lambda_1} \Rightarrow \gamma_1 \mid \Pi, \cdot(\Gamma_1) \Rightarrow \delta.$$

But also, easily

$$\vdash_{\text{CSemRL}} (\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Pi_1, \Gamma^{\lambda_1} \Rightarrow \gamma_1 \mid \Gamma_1 \Rightarrow \cdot(\Gamma_1).$$

So an application of (CUT) gives the desired result.

Now suppose that d_1 ends with:

$$\frac{\mathcal{G}' \mid \Pi_{1,1}, \Sigma_{1,1} \Rightarrow x \quad \mathcal{G}' \mid \Pi_{1,2}, \Sigma_{1,2}, [x]^{\lambda_1+\mu_1+1} \Rightarrow \gamma_1}{\mathcal{G}' \mid \Pi_1, [x]^{\lambda_1} \Rightarrow \gamma_1 \mid \Sigma_1, [x]^{\mu_1+1} \Rightarrow x} \ \text{(COM)}$$

where $\Pi_1 = \Pi_{1,1} \uplus \Pi_{1,2}$, $\Sigma_1 = \Sigma_{1,1} \uplus \Sigma_{1,2}$, and

$$\mathcal{G}' = \left([\Gamma_i \Rightarrow x]_{i=1}^n \mid [\Pi_j, [x]^{\lambda_j} \Rightarrow \gamma_j]_{j=2}^m \mid [\Sigma_k, [x]^{\mu_k+1} \Rightarrow x]_{k=2}^l\right).$$

By the induction hypothesis twice,

$$d_3 \vdash_{\mathsf{CSemRL}} \mathcal{H}' \mid \Pi_{1,1}, \Sigma_{1,1}, \Pi \Rightarrow \delta$$
$$\text{and} \quad d_4 \vdash_{\mathsf{CSemRL}} \mathcal{H}' \mid \Pi_{1,2}, \Sigma_{1,2}, \Gamma^{\lambda_1+\mu_1+1} \Rightarrow \gamma_1,$$

where $\mathcal{H}' = ((\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma, \Pi \Rightarrow \delta)$. So we can construct the following derivation:

$$\frac{\dfrac{\vdots\, d_3}{\mathcal{H}' \mid \Pi_{1,1}, \Sigma_{1,1}, \Pi \Rightarrow \delta} \quad \dfrac{\vdots\, d_4}{\mathcal{H}' \mid \Pi_{1,2}, \Sigma_{1,2}, \Gamma^{\lambda_1+\mu_1+1} \Rightarrow \gamma_1}}{\dfrac{\dfrac{\mathcal{H}' \mid \Gamma, \Pi \Rightarrow \delta \mid \Pi_1, \Sigma_1, \Gamma^{\lambda_1+\mu_1} \Rightarrow \gamma_1}{\mathcal{H}' \mid \Pi_1, \Sigma_1, \Gamma^{\lambda_1+\mu_1} \Rightarrow \gamma_1} \ \text{(EC)} \quad \overline{\mathcal{H}' \mid \Rightarrow \mathrm{e}} \ (\Rightarrow\mathrm{e})}{\mathcal{H}' \mid \Pi_1, \Gamma^{\lambda_1} \Rightarrow \gamma_1 \mid \Sigma_1, \Gamma^{\mu_1} \Rightarrow \mathrm{e}} \ \text{(COM)}} \ \text{(COM)}$$

Suppose that d_1 ends with

$$\frac{\mathcal{G}' \mid \Sigma_{1,1}, \Sigma_{2,1}, [x]^{\mu_1+\mu_2+2} \Rightarrow x \quad \mathcal{G}' \mid \Sigma_{1,2}, \Sigma_{2,2} \Rightarrow x}{\mathcal{G}' \mid \Sigma_1, [x]^{\mu_1+1} \Rightarrow x \mid \Sigma_2, [x]^{\mu_2+1} \Rightarrow x} \ \text{(COM)}$$

where $\Sigma_1 = \Sigma_{1,1} \uplus \Sigma_{1,2}$, $\Sigma_2 = \Sigma_{2,1} \uplus \Sigma_{2,2}$, and

$$\mathcal{G}' = ([\Gamma_i \Rightarrow x]_{i=1}^n \mid [\Pi_j, [x]^{\lambda_j} \Rightarrow \gamma_j]_{j=1}^m$$
$$\mid [\Sigma_k, [x]^{\mu_k+1} \Rightarrow x]_{k=3}^l).$$

By the induction hypothesis twice,

$$d_3 \vdash_{\mathsf{CSemRL}} \mathcal{H}' \mid \Sigma_{1,1}, \Sigma_{2,1}, \Gamma^{\mu_1+\mu_2+1} \Rightarrow \mathrm{e}$$
$$\text{and} \quad d_4 \vdash_{\mathsf{CSemRL}} \mathcal{H}' \mid \Sigma_{1,2}, \Sigma_{2,2}, \Pi \Rightarrow \delta$$

where $\mathcal{H}' = ((\mathcal{G}', \mathcal{H}_x)^{\mathcal{D}} \mid \Gamma, \Pi \Rightarrow \delta)$. We first apply the rule $(\mathrm{e}\Rightarrow)$ to last hypersequent of d_4, obtaining a derivation of $\mathcal{H}' \mid \Sigma_{1,2}, \Sigma_{2,2}, \Pi, \mathrm{e} \Rightarrow \delta$. Then by (CUT) with the last hypersequent of d_3, we obtain a derivation

$$d_5 \vdash_{\mathsf{CSemRL}} \mathcal{H}' \mid \Sigma_1, \Sigma_2, \Gamma^{\mu_1+\mu_2+1}, \Pi \Rightarrow \delta.$$

The required derivation is then

$$\frac{\dfrac{\dfrac{\vdots\, d_5}{\mathcal{H}' \mid \Sigma_1, \Sigma_2, \Gamma^{\mu_1+\mu_2+1}, \Pi \Rightarrow \delta} \quad \overline{\mathcal{H}' \mid \Rightarrow \mathrm{e}} \ (\Rightarrow\mathrm{e})}{\dfrac{\mathcal{H}' \mid \Gamma, \Pi \Rightarrow \delta \mid \Sigma_1, \Sigma_2, \Gamma^{\mu_1+\mu_2} \Rightarrow \mathrm{e}}{\mathcal{H}' \mid \Sigma_1, \Sigma_2, \Gamma^{\mu_1+\mu_2} \Rightarrow \mathrm{e}} \ \text{(EC)}} \ \text{(COM)} \quad \overline{\mathcal{H}' \mid \Rightarrow \mathrm{e}} \ (\Rightarrow\mathrm{e})}{\mathcal{H}' \mid \Sigma_1, \Gamma^{\mu_1} \Rightarrow \mathrm{e} \mid \Sigma_2, \Gamma^{\mu_2} \Rightarrow \mathrm{e}} \ \text{(COM)}$$

The remaining cases are all straightforward. □

We then obtain immediately from Lemma 4.

Theorem 8 *The variety of commutative semilinear residuated lattices is densifiable.*

5 Concluding remarks

The Jenei–Montagna method for proving densifiability has been used in Theorems 3, 4, and 5 of this paper to char-

acterize a broad range of densifiable varieties of semilinear residuated lattices, in particular, those defined over the variety of integral semilinear residuated lattices $\mathcal{SemIRL}$ by ℓ-monoid equations. This latter family may be further broadened using syntactic characterizations and proof-theoretic techniques based on the Metcalfe–Montagna method (Baldi and Ciabattoni 2015b; Baldi and Terui 2016). However, a general syntactic characterization of the densifiable varieties of integral semilinear residuated lattices that admit a cut-free hypersequent calculus is still lacking.

For non-integral varieties of semilinear residuated lattices, the picture is less clear. The proof-theoretic proof of densifiability described here for the variety $\mathcal{CSemRL}$ of commutative semilinear residuated lattices has been extended to other non-integral varieties in Metcalfe and Montagna (2007), Ciabattoni and Metcalfe (2008), Baldi (2014) and Baldi and Ciabattoni (2015a), but the scope of the method is unclear. It is not known, for example, if all varieties defined over $\mathcal{CSemRL}$ by monoid inequations are densifiable. Moreover, there are two specific cases that are of particular interest. First, it is not known if the variety of involutive commutative pointed semilinear residuated lattices, defined over pointed $\mathcal{CSemRL}$ by adding the involution axiom schema $\neg\neg x \approx x$,

is densifiable. Second, an axiomatization is lacking for the variety of residuated lattices generated by all dense residuated chains.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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