Commutative Residuated Lattices

James B. Hart, Lori Rafter, and Constantine Tsinakis

September 21, 2000

Abstract

A commutative *residuated lattice*, is an ordered algebraic structure $\mathbb{L} = (L, \cdot, \wedge, \vee, \rightarrow, e)$, where $(L, \cdot, e)$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and the operation $\rightarrow$ satisfies the equivalences

$$a \cdot b \leq c \iff a \leq b \rightarrow c \iff b \leq c \rightarrow a$$

for $a, b, c \in L$. The class of all commutative residuated lattices, denoted by $\mathcal{CRL}$, is a finitely based variety of algebras. Historically speaking, our study draws primary inspiration from the work of M. Ward and R. P. Dilworth appearing in a series of important papers [9] [10], [19], [20], [21] and [22]. In the ensuing decades special examples of commutative, residuated lattices have received considerable attention, but we believe that this is the first time that a comprehensive theory on the structure of residuated lattices has been presented from the viewpoint of universal algebra. In particular, we show that $\mathcal{CRL}$ is an "ideal variety" in the sense that its congruences correspond to order-convex subalgebras. As a consequence of the general theory, we present an equational basis for the subvariety $\mathcal{CRL}^{\mathrm{c}}$ generated by all commutative, residuated chains. We conclude the paper by proving that the congruence lattice of each member of $\mathcal{CRL}^{\mathrm{c}}$ is an

2000 *Mathematics Subject Classification* Primary 06B05; Secondary 06A15, 06B10, 06B20, 06D20, 06F05, 06F07, 06F20

algebraic, distributive lattice whose meet-prime elements form a root-system (dual tree). This result, together with the main results in [12] and [18], will be used in a future publication to analyze the structure of finite members of $\mathcal{CRL}^{c}$. A comprehensive study of, not necessarily commutative, residuated lattices will be presented in [4].

1 Preliminaries.

We begin with some notation. Let $(P, \leq)$ be a poset. We will use $\top$ and $\bot$ to denote the greatest and least elements, respectively, of P when such exist. Let $X \subseteq P$. We will let $\uparrow X$ denote the upperset generated by X in P; that is

$$\uparrow X = \{p \in P : x \leq p,\ \exists\ x \in X\}.$$

If $X = \{x\}$ is a singleton, we will use the symbol $\uparrow x$ in place of the more cumbersome $\uparrow \{x\}$. Uppersets of the form $\uparrow x$ will be called *principal*. The lowerset, $\downarrow X$, generated by X is defined dually.

A commutative binary operation $\cdot : P \times P \to P$ on a poset $(P, \leq)$ is said to be *residuated* provided there exists a binary operation $\to$: $P \times P \to P$ such that

$$a \cdot b \leq c \iff b \leq a \to c \iff a \leq b \to c.$$

In this event, we will say that $(P, \leq)$ is a *residuated* poset under the operation $\cdot$, and refer to the operation $\to$ as a *residual* of $\cdot$. Note that for all $a, b \in P$,

$$a \to b = max\ \{p \in P : a \cdot p \leq b\}.$$

The definition above serves as a natural extension of the notion of an adjoint pair in the sense that, for all $a \in P$, the assignment $x \mapsto a \to x$ serves as the right adjoint for the map $y \mapsto a \cdot y$. For an extensive discussion of adjunctions, see Gierz *et al.* [11].

As a consequence of the general theory of adjunctions, the binary operation $\cdot$ preserves all existing joins in both coordinates, and the residual preserves all existing meets in its second coordinate. Residuals of the operation $\cdot$ enjoy an additional property which will often prove useful. For any fixed $a \in P$, observe that

$$y \leq x \to a \iff x \cdot y \leq a \iff x \leq y \to a.$$

Hence, the operation $\to$ is dually self adjoint. Therefore, it is anti-isotone in its first component, converting all existing joins to meets (in P).

By a commutative *residuated lattice*, we shall mean an ordered algebraic structure of the form $\mathbb{L} = (L, \cdot, \wedge, \vee, \to, e)$, where

- $(L, \cdot, e)$ is a commutative monoid,
- $(L, \wedge, \vee)$ is a lattice, and
- the operation $\to$ serves as the residual for the monoid multiplication under the lattice ordering.

We denote the class of all such objects by $\mathcal{CRL}$.

We wish to advise the reader that we do not require the monoid identity to be the greatest element of the lattice. This constitutes a significant departure from the aforementioned work of Ward and Dilworth. See also Blyth [6] and McCarthy [16].

As an example, note that every Brouwerian algebra is a member of the class $\mathcal{CRL}$ where $\to$ in this case denotes the classical Brouwerian implication. It is worth noting that we are lax here with regard to the similarity type. Thus, we view a Brouwerian algebra as a member of the subvariety of $\mathcal{CRL}$ satisfying the additional identity $x \cdot y = x \wedge y$. Likewise, every abelian lattice-ordered group (ℓ-group) may be viewed as a member of the subvariety of $\mathcal{CRL}$ satisfying the additional identity $x \cdot (a \to e) = e$. In this case, $x \to e = x^{-1}$, and, more generally, $x \to y = x^{-1} \cdot y$. (For information on Brouwerian algebras, see Balbes and Dwinger [2] and Köhler [14]; for information on ℓ-groups, see Anderson and Feil [1].)

It is a routine matter to verify that a binary operation $\to$ is the residual of a commutative, associative operation $\cdot$ on a lattice $(L, \wedge, \vee)$ if and only if the following identities hold:

1. $x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z)$
2. $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$
3. $x \cdot (x \to y) \vee y = y$
4. $x \to (x \cdot y) \wedge y = y$

Thus, the class $\mathcal{CRL}$ is a variety.

We have mentioned two important subvarieties of $\mathcal{CRL}$, namely the subvariety of Brouwerian algebras and the subvariety of abelian ℓ-groups. Since the Brouwerian implication residuates the lattice meet operation, it is clear that any Brouwerian algebra is distributive. Likewise, it is well-known that the lattice reduct of any ℓ-group is also distributive (see for example Anderson and Feil [1]). It is certainly not necessary, however, for the lattice reduct of a residuated lattice to be distributive. There are a number of ways to construct nondistributive residuated lattices; one of the simplest ways was devised by Peter Jipsen and is presented below.

Example 1.1

Let $(L, \wedge, \vee, \bot, \top)$ be any bounded lattice containing an atom e. Define a binary operation $\cdot$ on L as follows.

$$x \cdot y = \begin{cases} \top & \text{if } x, y \notin \{\bot, e\} \\ x & \text{if } y = e \\ y & \text{if } x = e \\ \bot & \text{if } x = \bot \text{ or } y = \bot \end{cases}$$

It is easy to verify that the operation $\cdot$ is a multiplication on L with multiplicative identity e . It is also easy to verify that

$$x \to y = \begin{cases} \top & \text{if } x = \bot \text{ or } y = \top \\ e & \text{if } \bot < x \leq y < \top \\ \bot & \text{if } x \not\leq y \end{cases}$$

serves as the residual of the multiplication.

Among other things, the construction in Example 1.1 can be used to residuate any finite lattice, producing a distinct residutated multiplication for each atom.

We conclude this section by presenting a result that collects some basic properties of the arrow operation. Its simple proof is left to he reader.

Lemma 1.2 *Let $\mathbb{L}$ be a member of the variety $\mathcal{CRL}$. For all $a, b, c, d \in L$, the following statements are true:*

1. $e \to a = a$, *and* $e \leq a \to a$

2. $a \cdot (a \to b) \leq b$

3. $a \to (b \to c) = (a \cdot b) \to c$

4. $(a \to b) \to (c \to d) = c \to [(a \to b) \to d]$

5. $(a \to e) \cdot (b \to e) \leq (a \cdot b) \to e$.

□

2 Convex Subalgebras

The primary aim of this section is to establish that the congruence lattice of any member of $\mathcal{CRL}$ is isomorphic to the lattice of its convex subalgebras. In addition, we provide a canonical description of the elements of the convex subalgebra generated by an arbitrary subset. We begin with some terminology.

We will say that a subset C of a poset $(P, \leq)$ is *order-convex* (or simply convex) in P if, whenever $a, b \in C$, then $\uparrow a \cap \downarrow b \subseteq C$.

As usual, we will refer to a subset H of a commutative, residuated lattice $\mathbb{L}$ as being a *subalgebra* of $\mathbb{L}$ provided H is closed with respect to the operations defined on L. We will let $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$ denote the set of all convex subalgebras of $\mathbb{L}$, partially ordered by set-inclusion. It is easy to see that the intersection of any family of convex subalgebras of $\mathbb{L}$ is again a convex subalgebra of $\mathbb{L}$; hence, $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$ is a complete lattice in which meet is set-intersection.

In the work to follow, we will let $\mathtt{Con}(\mathbb{L})$ denote the lattice of congruence relations for a member $\mathbb{L}$ of the variety $\mathcal{CRL}$. Since the congruence lattice of any lattice, in particular that of the lattice-reduct of $\mathbb{L}$, is an algebraic distributive lattice, the same is true for $\mathtt{Con}(\mathbb{L})$. For all $\theta \in \mathtt{Con}(\mathbb{L})$, set

$$H_\theta = \{a \in L : (a, e) \in \theta\}.$$

For any convex subalgebra H of $\mathbb{L}$, set

$$\begin{aligned} \theta_H &= \{(a, b) \in L \times L : a \cdot h \leq b \text{ and } b \cdot h \leq a \ \exists\ h \in H\} \\ &= \{(a, b) \in L \times L : (a \to b) \wedge e \in H \text{ and } (b \to a) \wedge e \in H\} \end{aligned}$$

The first characterization of θ_H is the same as the one introduced by McCarthy in [16] for lattice-ordered semigroups. We shall use this characterization of θ_H and leave it to the reader to prove that the second characterization is equivalent. Our next goal will be to prove that the assignments $\theta \mapsto H_\theta$ and $H \mapsto \theta_H$ establish an order isomorphism between $\mathtt{Con}(\mathbb{L})$ and $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$.

Lemma 2.1. *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. If $\theta \in \mathtt{Con}(\mathbb{L})$, then H_θ is a convex subalgebra of $\mathbb{L}$.*

Proof. We first prove convexity. Suppose that $a, b \in H_\theta$, and suppose $x \in L$ is such that $a \leq x \leq b$. We must prove that $(x, e) \in \theta$.

Since $a \leq x$, we know $a \vee x = x$. Thus, since $(a, e), (x, x) \in \theta$, $(x, x \vee e) \in \theta$. Since $x \leq b$, $x = x \wedge b$. Thus, since $(b, e) \in \theta$, we see that $(x, (x \vee e) \wedge e) \in \theta$. However, since $e = e \wedge (e \vee x)$, we are done.

We now prove that H_θ is a subalgebra of $\mathbb{L}$. To this end, let $a, b \in H_\theta$. Since $(a, e), (b, e) \in \theta$, it follows at once that $(a \vee b, e), (a \wedge b, e)$, and $(a \cdot b, e)$ are all members of θ. Furthermore, since $e \to e = e$ by Lemma 1.2 (1), we see that $(a \to b, e) \in \theta$ as well.

□

Lemma 2.2. *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. If H is a convex subalgebra of $\mathbb{L}$, then θ_H is a congruence relation on $\mathbb{L}$.*

Proof. It is routine to prove that θ_H is an equivalence relation. To establish the substitution properties, let $(a, b), (c, d) \in \theta_H$. By definition, there exist $h, j \in H$ such that

- $a \cdot h \leq b$ and $b \cdot h \leq a$,
- $c \cdot j \leq d$ and $d \cdot h \leq c$.

Clearly, we may replace h and j by $k = h \wedge j$. Since the multiplication is isotone and preserves joins, it is easy to see that

- $(a \cdot c) \cdot k^2 \leq b \cdot d$ and $(b \cdot d) \cdot k^2 \leq a \cdot c$ and
- $(a \vee c) \cdot k \leq b \vee d$ and $(b \vee d) \cdot k \leq a \vee c$.

Consequently, $(a \cdot c, b \cdot d), (a \vee c, b \vee d) \in \theta_H$.

To see that $(a \wedge c, b \wedge d) \in \theta_H$, first observe that $a \leq k \to b$ and $c \leq k \to d$. Hence,

$$a \wedge c \leq (k \to b) \wedge (k \to d) = k \to (b \wedge d).$$

Thus, we have $(a \wedge c) \cdot k \leq (b \wedge d)$. The proof that $(b \wedge d) \cdot k \leq (a \wedge c)$ is similar.

It remains to prove that $(a \to c, b \to d) \in \theta_H$. Since $b \cdot k \leq a$, we know by Lemma 1.2 (2) that

$$(b \cdot k) \cdot (a \to c) \leq a \cdot (a \to c) \leq c.$$

Therefore, $(b \cdot k^2) \cdot (a \to c) \leq k \cdot c \leq d$, which implies

$$k^2 \cdot (a \to c) \leq b \to d.$$

The other inequality follows in like manner.

□

Theorem 2.3. *If $\mathbb{L}$ is a member of $\mathcal{CRL}$, then $\mathtt{Con}(\mathbb{L})$ is order isomorphic to $\mathcal{Sub}_{\mathcal{C}}(\mathbb{L})$. The isomorphism is established via the assignments $\theta \mapsto H_\theta$ and $H \mapsto \theta_H$.*

Proof. It is easy to see that the assignments are both isotone. It will therefore suffice to prove that $\theta_{H_\theta} = \theta$ and $H_{\theta_H} = H$.

Let $\theta \in \mathtt{Con}(\mathbb{L})$ and observe that

$$\theta_{H_\theta} = \{(x, y) \in L \times L : x \cdot h \leq y \text{ and } y \cdot h \leq x \ , \exists\ h \in H_\theta\}.$$

If $(a, b) \in \theta$, then $((a \to b) \wedge e, e) \in \theta$ and $((b \to a) \wedge e, e) \in \theta$ by Lemma 1.2 (1). Thus, if we let

$$h = (a \to b) \wedge (b \to a) \wedge e,$$

then it is clear that $h \in H_\theta$. Furthermore, we see by Lemma 1.2 (2) that

- $a \cdot h \leq a \cdot (a \to b) \leq b$, and
- $b \cdot h \leq b \cdot (b \to a) \leq a$.

Hence, we see that $(a, b) \in \theta_{H_\theta}$; it follows that $\theta \subseteq \theta_{H_\theta}$. On the other hand, suppose that $(x, y) \in \theta_{H_\theta}$. Then there exist $h \in L$ such that

- $(h, e) \in \theta$, and

- $x \cdot h \leq y$ and $y \cdot h \leq x$.

Consequently, we know $([x \cdot h] \vee y, x \vee y) \in \theta$ and $([y \cdot h] \vee x, x \vee y) \in \theta$. Transitivity and symmetry therefore imply

$$([y \cdot h] \vee x, [x \cdot h] \vee y) \in \theta.$$

Since $x = [x \cdot h] \vee x$ and $y = [y \cdot h] \vee y$, we see that $(x, y) \in \theta$. It follows that $\theta_{H_\theta} \subseteq \theta$.

Now, let $H \in \mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$ and observe that

$$H_{\theta_H} = \{a \in L : (a, e) \in \theta_H\}.$$

Suppose $a \in H$. Then $h = a \wedge (a \to e) \in H$; hence, we know $e \cdot h \leq e \cdot a \leq a$. By Lemma 1.2 (2), we also know $a \cdot h \leq a \cdot (a \to e) \leq a$. Hence, $(a, e) \in \theta_H$; and we see that $H \subseteq H_{\theta_H}$. On the other hand, suppose that $a \in H_{\theta_H}$. Then $(a, e) \in \theta_H$, which implies that there exists $h \in H$ such that

$$a \cdot h \leq e \text{ and } e \cdot h \leq a.$$

It follows that $h \leq a \leq h \to e$; hence, $a \in H$ by convexity.

□

We now turn attention to obtaining a simple internal characterization of convex subalgebras. Let $\mathbb{L}$ be a member of $\mathcal{CRL}$ and let H be any subalgebra of $\mathbb{L}$. Borrowing terminology from lattice-ordered groups, we will call the set $\downarrow e \cap H$ the *negative cone* of H. We will denote the negative cone of H by H^-. In symbols, we have

$$H^- = \{h \in H : h \leq e\}.$$

Corollary 2.4. *If $\mathbb{L}$ is a member of $\mathcal{CRL}$, then every convex subalgebra of $\mathbb{L}$ is completely determined by its negative cone. More specifically, if H is a convex subalgebra of $\mathbb{L}$, then*

$$H = \{x \in L : h \leq x \leq h \to e \text{ , } \exists\ h \in H^-\}.$$

Proof. Let H be a convex subalgebra of $\mathbb{L}$. Theorem 2.3 implies

$$\begin{aligned} H = H_{\theta_H} &= \{x \in L : (x, e) \in \theta_H\} \\ &= \{x \in L : x \cdot h \leq e \text{ and } h \leq x \ , \exists\ h \in H\} \\ &= \{x \in L : h \leq x \leq h \to e \ , \exists\ h \in H\} \\ &= \{x \in L : h \wedge e \leq x \leq (h \wedge e) \to e \ , \exists\ h \in H\} \\ &= \{x \in L : h' \leq x \leq h' \to e \ , \exists\ h' \in H^-\}. \end{aligned}$$

□

Note that the negative cone of any residuated lattice obtained by Example 1.1 contains only two elements. Hence by Corollary 2.4, any such algebra is *simple* in the sense that its congruence lattice contains exactly two members.

Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. In what follows, we will adopt a standard notation from semigroups. For all $n < \omega$, and for all $a \in L$, let

1. $a^0 = e$, and
2. $a^n = a \cdot a^{n-1}$ for all $0 < n < \omega$.

The following observations will be useful when dealing with members of the negative cone.

Lemma 2.5. *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. If a, b, c are members of the negative cone of $\mathbb{L}$, then the following are true:*

1. $a \leq a \to e$,
2. $(a \vee b) \cdot (a \vee c) \leq a \vee (b \cdot c)$,
3. *If $n < \omega$, then $(x \vee y)^n \leq x \vee y^n$,*
4. *If $m, n < \omega$, then $(x \vee y)^{mn} \leq x^m \vee y^n$.*

Proof. Claims (1) and (2) are obvious; we prove Claim (3) by means of induction. Since $a \leq e$, it is clear that Claim (3) holds when $n = 0$. Assuming the claim holds for any $n < \omega$, observe that

$$\begin{aligned}(a \vee b)^{n+1} &= (a \vee b) \cdot (a \vee b)^n \\ &\leq (a \vee b) \cdot (a \vee b^n) \\ &= a^2 \vee (a \cdot b^n) \vee (b \cdot a) \vee b^{n+1} \\ &\leq a \vee b^{n+1}.\end{aligned}$$

Claim (4) follows from Claim (3). Indeed, we have

$$(x \vee y)^{mn} = ((x \vee y)^n)^m \leq (x \vee y^n)^m \leq y^n \vee x^m.$$

□

For any $\mathbb{L}$ in $\mathcal{CRL}$ and any $S \subseteq L$, we will let $C[S]$ denote the smallest convex subalgebra of $\mathbb{L}$ containing S. As is customary, we will call $C[S]$ the convex subalgebra *generated* by S and will let $C[a] = C[\{a\}]$. We will call $C[a]$ the *principal* convex subalgebra of $\mathbb{L}$ generated by the singleton $\{a\}$ and usually refer to it as being generated by the element a. We will say that a convex subalgebra of $\mathbb{L}$ is *finitely generated* provided it is generated by some finite subset of L.

Lemma 2.6 *If $\mathbb{L}$ is a member of $\mathcal{CRL}$ and $S \subseteq L$, then $C[S] = C[S']$, where $S' = \{e \wedge a \wedge (a \to e) : a \in S\}$. In particular, every convex subalgebra of $\mathbb{L}$ is generated by a subset of its negative cone.*

Proof. Let $a \in L$ and $b = e \wedge a \wedge (a \to e)$. It will suffice to prove that $C[a] = C[b]$. First, note that b is clearly a member of the negative cone of $C[a]$; hence, $C[b] \subseteq C[a]$. Now, by Lemma 1.2 (2), we know

$$a \cdot b \leq a \cdot (a \to e) \leq e.$$

Since $b \leq a$ by construction, we see that $b \leq a \leq b \to e$; hence, $a \in C[b]$ by convexity.

□

Let $\mathbb{L}$ be a member of $\mathcal{CRL}$ and let $S \subseteq L$. In what follows, we will let $\langle S \rangle$ denote the *submonoid* of $(L, \cdot, e)$ generated by S.

Lemma 2.7. *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. If S is a subset of the negative cone of $\mathbb{L}$, then*

$$C[S] = \{x \in L : h \leq x \leq h \to e \ , \exists \ h \in \langle S \rangle\}.$$

Proof. If $S \subseteq L^-$, then clearly $\langle S \rangle \subseteq L^-$. Thus, if we let

$$K = \{x \in L : h \leq x \leq h \to e \ , \exists \ h \in \langle S \rangle\},$$

then it is clear that $S \subseteq K \subseteq C[S]$. It will therefore suffice to show that K is a convex subalgebra of $\mathbb{L}$. For all $a, b \in K$, note that there exist $h_a, h_b \in \langle S \rangle$ such that

$$h_a \leq a \leq h_a \to e \text{ and } h_b \leq b \leq h_b \to e.$$

Clearly, we may replace both h_a and h_b by $h = h_a \cdot h_b$. With this in mind, it is easy to see that K is convex. Indeed, if $a \leq x \leq b$, then $h \leq a \leq x \leq b \leq h \to e$; and we see that $x \in K$.

Likewise it is easy to verify that

$$h \leq a \wedge b \leq a \vee b \leq h \to e,$$

so K is closed with regard to meets and joins. Now, by Lemma 1.2 (5), we see at once that

$$h^2 \leq a \cdot b \leq (h \to e) \cdot (h \to e) \leq h^2 \to e.$$

Hence, K is also closed under multiplication. To see that K is closed under the arrow, first observe that $a \cdot h \leq e$ and $h \leq b$ together imply that $h^2 \leq a \to b$. On the other hand, $h \leq a$ implies

$$a \to b \leq h \to b \leq h \to (h \to e) = h^2 \to e$$

by Lemma 1.2 (3).

□

Corollary 2.8 *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$. If a is a member of the negative cone of $\mathbb{L}$, then*

$$C[a] = \{x \in L : a^n \leq x \leq a^n \to e \ , \exists \ n < \omega\}.$$

Our next goal will be to prove that every finitely generated convex subalgebra of a commutative, residuated lattice is principal.

Theorem 2.9. *If $\mathbb{L}$ is a member of $\mathcal{CRL}$, then $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$ is an algebraic, distributive lattice whose compact elements are the principal convex subalgebras of $\mathbb{L}$. Moreover, the compact elements of $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$ form a sublattice, with joins and meets given by the following formulas for all $a, b \in L^-$:*

$$C[a] \cap C[b] = C[a \vee b] \text{ and } C[a] \vee C[b] = C[a \wedge b].$$

Proof. We just need to establish the two equalities in the statement of the theorem. Let $a, b \in L^-$. Note that the inequalities $a, b \leq a \vee b \leq e$ imply that $a \vee b \in C[a] \cap C[b]$ and hence $C[a \vee b] \subseteq C[a] \cap C[b]$. To obtain the reverse inclusion, suppose $x \in C[a] \cap C[b]$. Then, by Corollary 2.8, there exist $m, n < \omega$ such

- $a^m \leq x \leq b^m \to e$, and
- $b^n \leq x \leq b^n \to e$.

Thus, by Lemma 2.5 (4), $(a \vee b)^{mn} \leq a^m \vee b^n \leq x$; and, likewise,

$$x \leq (a^m \to e) \wedge (b^n \to e) \leq (a \vee b)^{mn} \to e.$$

Hence, $x \in C[a \vee b]$. We have shown that $C[a] \cap C[b] = C[a \vee b]$.

Evidently, $a \wedge b \in C[a] \vee C[b] = C[\{a, b\}]$ Hence, $C[a \wedge b] \subseteq C[a] \vee C[b]$. To obtain the reverse inclusion, observe that, by Lemma 2.5 (1), we know

$$a \leq a \to e \leq (a \wedge b) \to e.$$

Thus, since $a \wedge b \leq a$, we see that $a \in C[a \wedge b]$. In like manner, we can prove that $b \in C[a \wedge b]$. It follows that $C[a] \vee C[b] \subseteq C[a \wedge b]$.

□

3 The Subvariety of $\mathcal{CRL}$ Generated by Residuated Chains

As mentioned at the beginning of Section 2, let $\mathcal{CRL}^{\mathrm{c}}$ denote the subvariety of $\mathcal{CRL}$ generated by residuated chains. Consider the following identities:

C_1 $e \leq (a \to b) \vee (b \to a)$

C_2 $e \wedge (a \vee b) = (e \wedge a) \vee (e \wedge b)$

Theorem 3.1 *The two identities above, together with those defining* $\mathcal{CRL}$*, form an equational basis for* $\mathcal{CRL}^{\mathrm{c}}$*.*

Proof. Let $\mathcal{V}$ denote the subvariety of $\mathcal{CRL}$ determined by identities C_1 and C_2. Since any commutative, residuated chain clearly satisfies these identities, it follows that every member of $\mathcal{CRL}^{\mathrm{c}}$ is a member of $\mathcal{V}$. To prove the converse, it will suffice to prove that every subdirectly irreducible member of $\mathcal{V}$ is totally ordered. We prove the contrapositive. Suppose that $\mathbb{L}$ satisfies identities C_1 and C_2 but is not totally ordered. Let a and b be incomparable elements in L. It follows that $e \not\leq a \to b$ and $e \not\leq b \to a$.

Let $u = e \wedge (a \to b)$ and let $v = e \wedge (b \to a)$. By choice of a and b, we infer that $e \neq u$ and $e \neq v$. By identity C_2, $u \vee v = e \wedge [(a \to b) \vee (b \to a)]$; hence, by identity C_1, $u \vee v = e$. However, by Theorem 2.9, this implies $C[u] \cap C[v] = \{e\}$. It follows that $\mathtt{Con}(\mathbb{L})$ cannot have a monolith. Thus $\mathbb{L}$ cannot be subdirectly irreducible.

□

The preceding result generalizes Theorem 4.4 in Cornish [8]. The class $\mathcal{CRL}^{\mathrm{c}}$ properly includes the subvarieties of generalized Boolean algebras, relative Stone algebras and commutative ℓ-groups. Interesting examples of members of $\mathcal{CRL}^{\mathrm{c}}$ are all ideal lattices of Dedekind domains (see Larsen *et al.* [15], p. 137), which served as the initial motivation for identities C_1 as well as identities E_1 and E_2 below.

Identities C_1 and C_2 must both be satisfied to guarantee that subdirectly irreducible members of $\mathcal{CRL}$ are totally ordered. To illustrate this, we consider two simple examples.

Example 3.2 Let $M_3 = \{\bot, a, b, e, \top\}$ denote the nondistributive diamond. The sublattice $B = M_3 - \{e\}$ is a Boolean lattice (and therefore a Brouwerian lattice). Define a binary operation $\cdot$ on M_3 as follows:

$$x \cdot y = \begin{cases} x \wedge y & \text{if } x, y \in B \\ x & \text{if } y = e. \end{cases}$$

This operation clearly defines an associative multiplication on M_3, with multiplicative identity e,which is residuated. If we let x^c denote the complement

of an element $x \in B$, then we can describe the residual for the multiplication as follows:

$$x \hookrightarrow y = \begin{cases} x^c \vee y & \text{if } x, y \in B \\ x^c & \text{if } x \in B \text{ and } y = e \\ y & \text{if } x = e \end{cases}$$

Now, the algebra $\mathbb{M}_3 = (M_3, \cdot, \wedge, \vee, \hookrightarrow, e)$ is simple and therefore subdirectly irreducible. Furthermore, it is easy to see that $\mathbb{M}_3$ satisfies identity C_1 but not identity C_2.

Example 3.3 On the other hand, consider the nonmodular pentagon $N_5 = \{\bot, a, b, e, \top\}$, where $\bot < a < b < \top$. The residuated lattice $\mathbb{N}_5$ obtained as in Example 1.1 is simple and therefore subdirectly irreducible. Moreover, it easily seen to satisfy identity C_2 (even though its lattice reduct is not distributive) but not identity C_1.

Consider the following identities:

E_1 $(a \wedge b) \to c = (a \to c) \vee (b \to c)$

E_2 $a \to (b \wedge c) = (a \to b) \vee (a \to c)$.

It is not difficult to verify that any member of $\mathcal{CRL}$ satisfying either E_1 or E_2 also satisfies C_1. Conversely, in view of Theorem 3.1, any lattice that satisfies C_1 and C_2 also satisfies E_1 and E_2, since the last two identities hold in residuated chains. In Theorem 3.4 below, we provide an elementary proof of this fact that does not rely on the Axiom of Choice. It is a slight variation of the proof given by Ward and Dilworth in [22] in the special case when e is the greatest element of the lattice.

Theorem 3.4. *Coupled with the identities defining $\mathcal{CRL}$, both $\{C_2, E_1\}$ and $\{C_2, E_2\}$ form alternative equational bases for $\mathcal{CRL}^{\mathrm{c}}$.*

Proof. We shall establish the equivalence of identities E_1 and E_2 when C_2 is satisfied without appealing to Theorem 3.1. As mentioned above, identities E_1 and E_2 each imply identity C_1. We first prove that identities C_1 and C_2 together imply identity E_1. Since the arrow is anti-isotone in its first component, it follows that

$$(b \wedge c) \to a \geq (b \to a) \vee (c \to a).$$

To obtain the reverse inequality, it will suffice to prove that

$$e \leq [(b \wedge c) \to a] \to [(b \to a) \vee (c \to a)].$$

Let $u = (b \to a) \vee (c \to a)$. Since the arrow is isotone in its second component, Lemma 1.2 (4) implies that

$$\begin{aligned} (b \wedge c) \to u &\geq [(b \wedge c) \to a] \to (b \to a) \\ &= b \to ([(b \wedge c) \to a] \to a) \\ &\geq b \to (b \wedge c), \end{aligned}$$

where the last inequality follows from Lemma 1.2 (2). Likewise,

$$\begin{aligned} (b \wedge c) \to u &\geq [(b \wedge c) \to a] \to (c \to a) \\ &= c \to ([(b \wedge c) \to a] \to a) \\ &\geq c \to (b \wedge c). \end{aligned}$$

Now, since the arrow preserves meets in its second component, it follows that

$$b \to (b \wedge c) = (b \to b) \wedge (b \to c \geq e \wedge (b \to c),$$

by Lemma 1.2 (1). Likewise, we know $c \to (b \wedge c) \geq e \wedge (c \to b)$. Therefore, it follows that

$$\begin{aligned} (b \wedge c) \to u &\geq [e \wedge (b \to c)] \vee [e \wedge (c \to b)] \\ &= e \wedge [(b \to c) \vee (c \to b)] \geq e. \end{aligned}$$

Note that we used identity C_2 for the last equality above, and we used identity C_1 to obtain the last inequality.

We now prove that identities C_1 and C_2 together imply identity E_2. Since the arrow operation is isotone in its second component, we know at once that

$$(a \to b) \vee (a \to c) \leq a \to (b \vee c).$$

To establish the reverse inequality, we will prove that

$$[a \to (b \vee c)] \to [(a \to b) \vee (a \to c)] \geq e.$$

Let $u = (a \to b) \vee (a \to c)$. Since the arrow is isotone in its second component, we have by Lemma 1.2 (4), (3), and (2) that

$$\begin{aligned} [a \to (b \vee c)] \to u &\geq [a \to (b \vee c)] \to (a \to b) \\ &= a \to [\{a \to (b \vee c)\} \to b] \\ &= [a \cdot (a \to (b \vee c))] \to b \\ &\geq (b \vee c) \to b. \end{aligned}$$

Likewise, we have $[a \to (b \vee c) \to u \geq (b \vee c) \to c$. Consequently,

$$\begin{aligned} [a \to (b \vee c)] \to u &\geq [(b \vee c) \to b] \vee [(b \vee c) \to c] \\ &= [(b \to b) \wedge (c \to b)] \vee [(c \to c) \wedge (b \to c)] \\ &\geq [e \wedge (c \to c)] \vee [e \wedge (b \to c)] \\ &= e \wedge [(c \to b) \vee (b \to c)] \geq e. \end{aligned}$$

Note that we used identity C_2 to obtain the last equality and used identity C_1 to obtain the last inequality.

□

In view of Theorem 3.1 and Theorem 3.4, we know that in the presence of identity C_2, the identities E_1, E_2, and C_1 are equivalent and imply distributivity. It is possible, however, to prove this implication directly, without appealing to the Axiom of Choice, as we now show. Again, the proof is a slight variation of the proof given by Ward and Dilworth in [22] in the special case when e is the greatest element of the lattice.

Theorem 3.5 *Let $\mathbb{L}$ be a member of $\mathcal{CRL}$ satisfying identity C_2. If $\mathbb{L}$ satisfies the equivalent identities E_1, E_2, and C_1, then its lattice reduct is distributive.*

Proof. It suffices to prove that, for all $a, b, c \in L$, $a \leq (b \vee c)$ implies $a = (a \wedge b) \vee (a \wedge c)$. We need only establish that $(a \wedge b) \vee (a \wedge c) \leq a$; the other inequality clearly holds in any lattice.

First, observe that, for all $a, b, c \in L$, identity E_1 implies

$$a \to [(a \wedge b) \vee (a \wedge c)] = [a \to (a \wedge b)] \vee [a \to (a \wedge c)].$$

Since the residual preserves meets in its second argument, we know by Lemma 1.2 (1) that

$$a \to [(a \wedge b) \vee (a \wedge c)] \geq [e \wedge (a \to b)] \vee [e \wedge (a \to c)].$$

Hence, Identities C_2 and E_2 together imply

$$a \to [(a \wedge b) \vee (a \wedge c)] \geq e \wedge [a \to (b \vee c)]$$

.

Now, if we assume that $a \leq (b \vee c)$, then $a \to (b \vee c) \geq e$. Consequently, if $a \leq b \vee c$, we see that

$$a \to [(a \wedge b) \vee (a \wedge c)] \geq e,$$

which, of course, implies that $a \leq (a \wedge b) \vee (a \wedge c)$.

□

We note that in the presence of identity C_2, the equivalent to identities C_1, E_1, and E_2 imply the identity $a \cdot (b \wedge c) = a \cdot b \wedge a \cdot c$. The converse is not true. For example, multiplication distributes over finite intersections in the ideal lattice of a Prüfer domain (see Larsen *et al.* [15], p. 137); however, not every such domain is a Dedekind domain.

4 More About the Structure of $\mathtt{Con}(\mathbb{L})$

In Section 2, we provided an element-wise description for the congruences of any member of $\mathbb{L}$ of $\mathcal{CRL}$ by establishing an isomorphism between $\mathtt{Con}(\mathbb{L})$ and $\mathtt{Sub}_{\mathcal{C}}(\mathbb{L})$. In the last section, we used this correspondence to characterize in terms of identities those members of $\mathcal{CRL}$ which are subdirect products of chains. We conclude this paper by proving that members of the variety $\mathcal{CRL}^{\mathrm{c}}$ enjoy an additional property — their compact congruences form relatively normal lattices.

A lattice $L = (L, \wedge, \vee)$ is *relatively normal* provided

1. L is a lower-bounded, distributive lattice, and

2. the prime-ideals of L form a root-system under set-inclusion.

Recall that a poset is a *root-system* provided its principal uppersets are chains. Relatively normal lattices form an important class of lattices in their own right. Dual relative Stone lattices and the lattice of open sets of any hereditarily normal topological space are relatively normal, as are the lattices of compact congruences for many well-studied ordered algebraic structures such as representable ℓ-groups, Reisz spaces, f-rings, and Stone algebras. (The reader is referred to Snodgrass and Tsinakis [18] and Hart and Tsinakis [12] for details and an extensive bibliography on this class.)

Since the introduction of this class in the 1950's, a number of conditions on lower-bounded, distributive lattices equivalent to relative normality have been obtained. The next result, due essentially to Monteiro [17], catalogues several of these conditions. Its proof is left to the reader. For our purposes, we shall find Condition (4) most useful.

Lemma 4.1 *For a lower-bounded, distributive lattice L, the following are equivalent:*

1. *L is relatively normal;*
2. *Any two incomparable prime ideals in L have disjoint open neighborhoods in the Stone space of L;*
3. *The join of any pair of incomparable prime filters in L is all of L;*
4. *For all $a, b \in L$, there exist $u, v \in L$ such that $u \wedge v = \bot$ and $u \vee b = a \vee b = a \vee v$.*

We hasten to advise the reader that many authors who consider relative normality express it as a property of the *closed* sets of a topological space (whereas we have used open sets); they will use and prove the duals of the statements in Lemma 4.1.

Theorem 4.2 *If $\mathbb{L}$ be a member of $\mathcal{CRL}^{\mathrm{c}}$, then the principal convex subalgebras of $\mathbb{L}$ form a relatively normal lattice.*

Proof. Throughout the proof, we will use Theorem 2.9 repeatedly without explicit reference. The compact, convex subalgebras of $\mathbb{L}$ form a lower-bounded, distributive lattice. Let $\mathcal{C}(\mathbb{L})$ denote this lattice, and let $X, Y \in \mathcal{C}(\mathbb{L})$. There exist a, b in the negative cone of $\mathbb{L}$ such that $X = C[a]$ and $Y = C[b]$. Let u and v be defined by

- $u = e \wedge (b \to a)$
- $v = e \wedge (a \to b)$.

We know know that $C[u] \cap C[v] = C[u \vee v]$. Consequently, since

$$u \vee v = e \wedge [(a \to b) \vee (b \to a)] = e,$$

by identities C_2 and C_1, we see that $C[u] \cap C[v] = \{e\}$. To complete the proof, we need to show that

$$C[a] \vee C[v] = C[a] \vee C[b] = C[u] \vee C[b].$$

We will prove that $C[a] \vee C[v] = C[a] \vee C[b]$; proof of the other equality is similar.

To prove that $C[a] \vee C[v] \subseteq C[a] \vee C[b]$, it suffices to prove that $a \wedge v \in C[a] \vee C[b]$. Since $a \leq e$ by assumption, we know $a \wedge v = a \wedge (a \to b)$. By definition, $C[a] \vee C[b] = C[\{a, b\}]$; hence, it is clear that $a \wedge v \in C[a] \vee C[b]$.

To obtain the reverse inclusion, it suffices to prove that $a \wedge b \in C[a] \vee C[v]$. Since multiplication distributes over finite meets (see comments at the end of Section 2), we have

$$(a \wedge v)^2 = a^2 \wedge [a \cdot (a \to b)] \wedge (b \to a)^2.$$

Since $a^2 \leq a$ by assumption, and since $a \cdot (a \to b) \leq b$ by Lemma 1.2 (2), it follows that $(a \wedge v)^2 \leq a \wedge b$. Since $a \wedge b \leq e$ by assumption, convexity now implies that $a \wedge b \in C[a] \vee C[v]$.

□

References

[1] M. Anderson and T. Feil, *Lattice-Ordered Groups, An Introduction*, Reidel, Dordrecht, Holland, 1988.

[2] R. Balbes and P. Dwinger, *Distributive Lattices*, Columbia, University of Missouri Press, 1974

[3] Garrett Birkhoff, *Lattice Theory*, volume 25 of *Colloquium Publications*, American Mathematical Society, 1967.

[4] K. Blount and C. Tsinakis, *Residuated Lattices*, in preparation.

[5] T. S. Blyth, The general form of residuated algebraic structures, *Bull. Soc. Math. France*, 93 (1965), pp. 109-167.

[6] T. S. Blyth, Loipomorphisms, *J. London Math. Soc.* (2), 2 (1970), 635-642.

[7] T.S. Blyth and M.F. Janowitz, *Residuation Theory*, Pergamon Press, 1972.

[8] W.H. Cornish, Subdirect Decompositions of Semilattice-ordered semigroups,*Mathematica Japonica*, 18 (1973), 203-209.

[9] R.P. Dilworth, Abstract residuation over lattices, *Bull. Amer. Math. Soc.*, 44 (1938), 262-268.

[10] R.P. Dilworth, Non-commutative residuated lattices, *Trans. Amer. Math. Soc.*, 46 (1939), 426-444.

[11] G. Gierz, K. Hoffmann, K. Keimel, J. Lawson, M. Mislove, and D. Scott, *A Compendium of Continuous Lattices*, Springer-Verlag, Berlin Heidleberg New York, 1980.

[12] J.B. Hart and C. Tsinakis, Decompositions for relatively normal lattices, *Trans. Amer. Math. Soc.*, 341 (1994), 519 - 548.

[13] A. Horn, Logic with truth values in a linearly ordered Heyting algebra, *J. Symbolic Logic*, 34 (1969), 395-408.

[14] P. Köhler, *Varieties of Brouwerian algebras*, Mitt. Math. Sem. Geissen. Selbstverlag des mathematischen seminars, 1975.

[15] M. Larsen and P. McCarthy, *Multiplicative Theory of Ideals*, Academic Press New York and London, 1971.

[16] P. J. McCarthy, Homomorphisms of certain commutative lattice ordered semigroups,*Acta Sci. Math. (Szeged)* 27 (1966), 63-65.

[17] A. Monteiro, L'arithmetique des filtres et les espaces topologiques, *De Segundo Symposium de Matematicas-Villavicencio* (Mendoza, Buenos Aires) (1954), 129-162.

[18] J. Snodgrass and C. Tsinakis, Finite-valued algebraic lattices, *Algebra Universalis* 30 (1993), 311-318.

[19] M. Ward, Structure Residuation, *Annals of Math.*, 39 (1938), 558-568.

[20] M. Ward, Residuation in structures over which a multiplication is defined, *Duke Math. Journal*, 3 (1937), 627-636.

[21] M. Ward and R. P. Dilworth, Residuated lattices, *Proceedings of the National Academy of Sciencs*, 24 (1938), 162-164.

[22] M. Ward and R. P. Dilworth, Residuated lattices, *Trans. Amer. Math. Soc.*, 45 (1939), 335-354.

---

1. James Hart, Box 34, Middle Tennessee State University, Murfreesboro TN 37132, (615)-898-2669, FAX (615) 898-5422, email: *jhart@mtsu.edu*

2. Lori Rafter, Department of Mathematics, Vanderbilt University, Nashville TN 37235, (615) 322-6672

3. Constantine Tsinakis, Department of Mathematics, Vanderbilt University, Nashville TN 37235, (615) 322-6672, FAX (615) 343-0215, email: *constantine.tsinakis@vanderbilt.edu*