Ordinal Decompositions for Preordered Root Systems

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Abstract

In this paper, we explore the effects of certain forbidden substructure conditions on preordered sets. In particular, we characterize in terms of these conditions those preordered sets which can be represented as the supremum of a well-ordered ascending chain of lowersets whose members are constructed by means of alternating applications of disjoint union and ordinal sums with chains. These decompositions are examples of *ordinal decompositions* in relatively normal lattices as introduced by Snodgrass, Tsinakis, and Hart. We conclude the paper with an application to information systems.

1 Introduction

The term "ordinal decomposition" refers to an inductive construction introduced by Snodgrass, Tsinakis, and Hart [6, 12, 13] that traces its origins to the lexicographic sums of lattice-ordered groups explored in [2]. The paper Tsinakis and Hart [6] defines these constructions, addresses existence and uniqueness of ordinal decompositions, and presents motivation for their study. The key result of that paper establishes that the most general context for studying ordinal decompositions is a subclass of *relatively normal lattices* (lower-bounded distributive lattices whose prime ideals form a root system under set-inclusion) delineated by certain forbidden substructures. Relatively normal lattices form a diverse class that includes the lattices of compact congruences (principal convex normal subalgebras) of representable residuated lattices. The variety of representable residuated lattices is generated by its totally ordered members and includes the varieties of MV-algebras, BL-algebras and Gödel algebras.

In this paper, we consider order structures more primitive than, but still tied to, relatively normal lattices; namely preordered root systems. (The finitely generated lowersets of such systems form relatively normal lattices under set inclusion — see Snodgrass and Tsinakis [12], Theorem 2.7). We demonstrate that the notion of an ordinal decomposition arises quite naturally from a simple classification scheme for posets satisfying the descending chain condition. We conclude the paper by using the decomposition theory developed in Sections 2 and 3 below to explore the structure of information systems, important objects in theoretical computer science.

An important potential application of our considerations to algebraic logic – which will not be pursued here – is the development of a process for building a large class of algebras in the varieties of BL-algebras and Gödel algebras by starting with their totally ordered members and making use of the ordinal decomposition of their filter lattice.

2 The Skeleton of a Preordered Root System

A preordered set $R = (R, \leq)$ is a *root system* (dual tree) provided the principal upperset $\uparrow r = \{x \in R : r \leq x\}$ is a preordered chain for all $r \in R$. In this paper, we will develop a means of representing certain preordered root systems as the suprema of well-ordered ascending chains of inductively constructed lowersets. These decompositions are straightforward, are uniquely determined by the order structure of the root system, and can be applied in many areas of order theory. Before presenting these decompositions in their full generality, however, we will devote this section to a special case which provides the basis for, and an explanation of, their construction.

Let R be a nonempty preordered root system which satisfies the descending chain condition (DCC). Under these conditions, there is a straightforward way to organize the elements of R. Let τ be an ordinal and let $F_\tau = \{B_\sigma : \sigma < \tau\}$ be a collection of maximal antichains in R constructed as follows:

1. B_0 consists of all minimal elements of R.
2. If $\sigma < \tau$ has an immediate predecessor, then $r \in B_\sigma$ if and only if
   (a) r is a minimal element of $R \setminus \bigcup\{B_\eta : \eta < \sigma\}$, or
   (b) $r \in B_{\sigma-1}$ and $\uparrow r$ contains no minimal elements of $R \setminus B_{\sigma-1}$.
3. If $0 < \sigma < \tau$ is a limit ordinal, then B_σ denotes the set of all minimal elements of $R \setminus \bigcup\{B_\eta : \eta < \sigma\}$.

For reference purposes, we will call this process the *classification scheme* for R, and will refer to the sets B_σ as the σ-levels of this scheme. The 0-level will be called the initial level of the scheme.

The classification scheme we have just described can be accomplished in any preordered set which satisfies DCC. However, the root system property gives us a simple relationship between members of the sets B_σ:

- If σ has an immediate predecessor, then every member of B_σ must cover at least one member of $B_{\sigma-1}$;
- If $0 < \sigma$ is a limit ordinal, then for each $b \in B_\sigma$ there exists an ascending chain $C = \{b_\eta : \eta < \sigma, b_\eta \in B_\eta\}$ and $\downarrow b$ covers $\downarrow C$ in $O(R)$, the lattice of lowersets of R.

Lemma 2.1. *If R is a nonempty preordered root system satisfying DCC, then there exists an ordinal τ such that $R = \bigcup\{B_\eta : \eta < \tau\}$.*

Proof. Since R is nonempty, we know that $B_0 \neq \emptyset$. Let $0 < \sigma < \tau$ and suppose that there exist $r \in R$ such that $r \notin B_\eta$ for all $\eta < \sigma$. We will prove that the set B_σ can be constructed, and that $B_\sigma \neq B_\eta$ for all $\eta < \sigma$.

Let S denote the set of all members of $\downarrow r$ which do not reside in B_η for any $\eta < \sigma$. Clearly, $S \neq \emptyset$ and therefore must contain minimal elements. Let $s \in S$ be minimal in S. We will prove that $s \in B_\sigma$.

Since $s \notin B_0$, we know that s is not minimal in R; hence, there exist $t \in R$ such that $t < s$. Now, by assumption, for each $t < s$, there exist $\eta_t < \sigma$ such that $t \in B_{\eta_t}$. Thus, s is minimal in $R \setminus \bigcup\{B_\eta : \eta < \sigma\}$. Consequently, $s \in B_\sigma$. ☐

Lemma 2.1 is valid only in root systems which satisfy DCC. It is possible, however, to extend this result to a considerably broader class of preordered sets; and, in so doing, develop a useful means of decomposing members of this class. We begin with a definition.

Definition 2.2. *Let P be a preordered set and let $r, s \in P$. We say that r is an ordinal extension of s provided*

1. *$s \in \downarrow r$*
2. *$\downarrow r \setminus \downarrow s$ is a preordered chain, every member of which is comparable to s.*

Definition 2.3. *Let R be any preordered root system. Define a binary relation* `Ord` *on R as follows: $(r, s) \in$* `Ord` *if and only if*

- *r is an ordinal extension of s, or*
- *s is an ordinal extension of r.*

Since R is a root system, it is routine to prove that the sets

$$\mathtt{Ord}(r) = \{s \in R : (r, s) \in \mathtt{Ord}\}$$

are pairwise disjoint. Hence, `Ord` is an equivalence relation. Now, for $\mathtt{Ord}(r), \mathtt{Ord}(s) \in R/\mathtt{Ord}$, set $\mathtt{Ord}(r) \preceq \mathtt{Ord}(s)$ if and only if

1. $\mathtt{Ord}(r) = \mathtt{Ord}(s)$ or
2. $\mathtt{Ord}(r) \neq \mathtt{Ord}(s)$ and $r \leq s$.

Once again, since R is a root system, it is routine to prove that $\preceq$ is well-defined on $R/\mathtt{Ord}$ and is, in fact, a partial ordering.

By equating each member of $R/\mathtt{Ord}$ with one of its representatives, we can identify $R/\mathtt{Ord}$ with a subposet of R. We will let $\mathtt{Skel}(R)$ denote any one of these subposets (all of which are, of course, order isomorphic). We will call $\mathtt{Skel}(R)$ a *skeleton* for R. Observe that $\mathtt{Skel}(R)$ is a partially ordered root system.

Definition 2.4. *Let $P = (P, \leq)$ be a preordered set and let S be a subpreordered set of P. We will say that S is a filet configuration in P provided*

$$S = \{a_n : n < N\} \cup \{b_{n+1} : n < N\}$$

where $N \leq \omega$ and

- *a_n is an upper bound for a_{n+1} and b_{n+1};*
- *a_{n+1} is not comparable to b_{n+1}.*

We refer to the ordinal N as the *height* of the filet configuration. The members a_n of the configuration are called *filets*. A filet configuration is infinite when $N = \omega$. The notion of filet configuration dates to Jaffard [7] where it appeared in the context of lattice-ordered groups (see also McAlister [8] and Hart and Tsinakis [6]).

Lemma 2.5. *If R is a preordered root system, then the following are equivalent:*

1. *Every filet configuration in R is finite;*
2. *$R/\mathtt{Ord}$ satisfies DCC.*

Proof. Every descending chain of height N in $R/\mathtt{Ord}$ induces a filet configuration of height N in R. To see why, suppose $C = \{\mathtt{Ord}(a_n) : n < N\}$ is a descending chain of height N in $R/\mathtt{Ord}$. Since $\mathtt{Ord}(a_{n+1}) \prec \mathtt{Ord}(a_n)$, the root system property of R implies that there exist b_{n+1} such that a_{n+1} is not comparable to b_{n+1} and $b_{n+1} < a_n$. The set $\{a_n : n < N\} \cup \{b_{n+1} : n < N\}$ is a filet configuration of height N in R. It now follows that if every filet configuration in R is finite, then $R/\mathtt{Ord}$ must satisfy DCC.

Conversely, every filet configuration in R of height N induces a descending chain of height N in $R/\mathtt{Ord}$. To see why, let $S = \{a_n : n < N\} \cup \{b_{n+1} : n < N\}$ be a filet configuration of height N and consider the classes $\mathtt{Ord}(a_n)$ in $R/\mathtt{Ord}$. Since $b_{n+1} < a_n$, $a_{n+1} < a_n$, and b_{n+1} is not comparable to a_{n+1}, it follows that $\mathtt{Ord}(a_{n+1}) \prec \mathtt{Ord}(a_n)$ for all $n < N$. Consequently, the set $\{\mathtt{Ord}(a_n) : n < N\}$ forms a descending chain in $R/\mathtt{Ord}$ of height N. It now follows that if $R/\mathtt{Ord}$ satisfies DCC, then every filet configuration in R must be finite. □

We are now ready to prove the main result of this section: Whenever we have a root system R in which every filet configuration is finite, we may apply our classification scheme to $\mathtt{Skel}(R)$ to obtain a decomposition of R in terms of inductively defined members of $O(R)$, the lattice of lowersets of R. More specifically, we have the following result.

Theorem 2.6. *Let R be a preordered root system in which every filet configuration is finite. Then there exists an ordinal τ and an ascending chain $C = \{R_\sigma : \sigma < \tau\}$ of lowersets of R such that $R = \bigcup C$, and each $R_\sigma = \bigcup\{\mathtt{Ord}(a) : a \in B_\eta$ for some $\eta \leq \sigma\}$, where B_η is the η-level of the classification scheme for $\mathtt{Skel}(R)$.*

Proof. In light of Lemmas 2.1 and 2.5, we need only prove that the sets R_σ are lowersets of R. Let $R_\sigma \in C$, $r \in R$, and suppose that $r \leq x$ for some $x \in R_\sigma$. Let $a, b \in \mathtt{Skel}(R)$ correspond to $\mathtt{Ord}(r)$ and $\mathtt{Ord}(x)$, respectively. Since $x \in R_\sigma$, we know $b \in B_\eta$ for some $\eta < \sigma$. Clearly, $a \leq b$; consequently, there exists an ordinal $\eta \leq \sigma$ such that $a \in B_\eta$. Thus, $r \in R_\sigma$. □

3 Ordinal Decompositions

In this section, we will examine the implications of Theorem 2.6. In particular, we will obtain a description of the decomposition which can be liberated from its dependence on the root system property. We begin with some terminology.

Let P be a preordered set and suppose $r, s \in P$. Whenever r is an ordinal extension of s and $\downarrow r \setminus \downarrow s \neq \emptyset$, we say r is a *proper* ordinal extension of s and refer to r as an *ordinal element* in P. An ordinal element r of P is a *maximal* ordinal element in P provided r admits no proper ordinal extension. We will use $\mathtt{M}(P)$ to denote the sub-preordered set of maximal ordinal elements of P.

Proposition 3.1. *Let R be a preordered set. The ordinal elements of $O(R)$ form a root system if and only if R is a root system.*

Proof. Clearly, $\downarrow r$ is an ordinal element in $O(R)$ for all $r \in R$; hence the ordinal elements of $O(R)$ cannot form a root system if R is not a root system. Conversely, suppose R is a root system, and suppose that X, Y are incomparable ordinal elements in $O(R)$. Let $a \in X$ and $b \in Y$ be such that a and b are incomparable in R. Since R is a root system, we know that $\{a, b\}$ cannot have a lower bound in R. Hence, $\downarrow a$ and $\downarrow b$ are disjoint. Therefore, X and Y must be disjoint. It follows that the ordinal elements of $O(R)$ form a root system. □

Theorem 3.2. *Let R be a preordered root system. The set $M(O(R))$ of maximal ordinal elements of $O(R)$ is order isomorphic to $R/\mathtt{Ord}$.*

Proof. Let $a \in R$ and consider $\downarrow\mathtt{Ord}(a)$. If $Y = \downarrow\mathtt{Ord}(a) \setminus \mathtt{Ord}(a)$, then it is clear that $\downarrow a$ is a proper ordinal extension of Y in $O(R)$. Hence, $\downarrow\mathtt{Ord}(a)$ is an ordinal element in $O(R)$. Now, suppose J is any ordinal extension of $\downarrow\mathtt{Ord}(a)$. If there exist $j \in J \setminus \downarrow\mathtt{Ord}(a)$, then j must be an ordinal extension of a in R. This implies $j \in \mathtt{Ord}(a)$ — contrary to assumption. Hence, we must have $J = \downarrow\mathtt{Ord}(a)$; and $\downarrow\mathtt{Ord}(a)$ is a maximal ordinal element in $O(R)$. In light of this discussion, we may define a mapping $f : R/\mathtt{Ord} \longrightarrow M(O(R))$ by $f(\mathtt{Ord}(a)) = \downarrow\mathtt{Ord}(a)$.

On the other hand, suppose that $M \in M(O(R))$. It follows that M is a proper ordinal extension of some $X \in O(R)$. Let $j \in \bigcup M \setminus X$, and consider $\mathtt{Ord}(j)$. Clearly, we must have $\mathtt{Ord}(j) \subset M$ by Proposition 3.1. Let $Y = M \setminus \mathtt{Ord}(j)$. Clearly, M is a maximal ordinal extension of Y in $O(R)$; hence, $M = \downarrow\mathtt{Ord}(j)$ by Proposition 3.1. In light of this discussion, we may define a mapping $g : M(O(R)) \longrightarrow R/\mathtt{Ord}$ by $g(M) = \mathtt{Ord}(j)$, where $j \in M$ is such that M is a proper ordinal extension of $\downarrow j$.

The mappings f and g are clearly mutually inverse and isotone, and therefore provide the desired isomorphism. □

Let R be a preordered root system in which every filet configuration is finite. Let us return to Theorem 2.6 and consider more carefully the lowersets R_σ of the decomposition described there. It is possible to represent each R_σ as the union of a maximal pairwise disjoint set whose members are either maximal ordinal elements in $O(R)$ or else the union of a transfinite chain of such elements. This fact is a direct consequence of our classification scheme for Skel(R) and Theorem 3.2, as we now demonstrate.

If R is a preordered root system in which every filet configuration is finite, then Theorem 2.6 tells us there exists an ordinal τ and an ascending chain $C = \{R_\sigma : \sigma < \tau\}$ such that $R = \bigcup C$, where

$$R_\sigma = \bigcup\{\mathtt{Ord}(a) : a \in B_\eta \text{ for some } \eta \leq \sigma\}$$

and B_η is the η-level of the classification scheme for Skel(R). By definition, $R_0 = \bigcup\{\mathtt{Ord}(a) : a \in B_0\}$. Since B_0 is the set of all minimal elements of Skel(R), we know $\mathtt{Ord}(a) = \downarrow\mathtt{Ord}(a)$ in R; hence, by Theorem 3.2, the set $C_0 = \{\mathtt{Ord}(a) : a \in B_0\}$ is a maximal set of pairwise disjoint maximal ordinal elements. In fact, C_0 consists entirely of maximal *linear* elements in $O(R)$; that is, maximal lowersets which are chains.

Consider R_n, where $0 < n < \omega$. By definition,

$$R_n = \bigcup\{\mathtt{Ord}(a) : \mathtt{Ord}(a) \in B_0 \cup ... \cup B_n\}$$

Let $0 < j \leq n$. According to our classification scheme, each member of B_j is either a member of B_{j-1} or else exceeds a member of B_{j-1}. Consider the set $C_n = \{\downarrow\mathtt{Ord}(a) : a \in B_n\}$. Clearly, $R_n = \bigcup C_n$; and, since B_n is by construction a maximal antichain in Skel(R), we know by Theorem 3.2 that C_n is a maximal, pairwise disjoint set of maximal ordinal elements in $O(R)$.

If we consider the sequence $C_0, ..., C_{n-1}$ so constructed, we can say even more about the members of C_n: A member I of C_n is either a member of C_{n-1}, or else is the maximal ordinal extension in $O(R)$ of the union of two or more members of C_{n-1}. Indeed, suppose $I \in C_n$ but $I \notin C_{n-1}$. It follows that I properly contains a member of C_{n-1}. In fact, since no member of $\mathtt{Skel}(R)$ can be a proper ordinal extension of another, we know that I properly contains at least two members of C_{n-1}. If we let D_I denote the set of all members of C_{n-1} (properly) contained in I, then I must be a proper ordinal extension of $\bigcup D_I$. To see why, let $I = \downarrow \mathtt{Ord}(a)$, where $a \in \mathtt{Skel}(R)$ and suppose $x \in I \setminus \mathtt{Ord}(a)$. Since $x \in R_n$, there must exist $b < a$ such that $x \in \mathtt{Ord}(b)$. It follows that there must exist $J \in D_I$ such that $x \in J$; hence, $x \in \bigcup D_I$. Consequently, $I \setminus \bigcup D_I = \mathtt{Ord}(a)$; and I is a (maximal) ordinal extension of $\bigcup D_I$.

Consider now the lowerset R_ω. By definition, we know

$$R_\omega = \bigcup \{\mathtt{Ord}(a) : a \in B_\eta \text{ for some } \eta \leq \omega\}$$

Let $D_\omega = \{\bigcup A^i : i \in I\}$, where each $A^i = \{J_n^i : n < \omega\}$ is an ascending chain such that $J_n^i \in C_n$ for each $n < \omega$. It is routine to prove that D_ω consists of pairwise disjoint lowersets. Let $C'_\omega = \{\downarrow \mathtt{Ord}(a) : a \in B_\omega\}$. According to our classification scheme for $\mathtt{Skel}$, each member of C'_ω must properly contain a member of D_ω. Let D^*_ω denote the set of all members of D_ω which are not contained in a member of C'_ω, and let $C_\omega = C'_\omega \cup D^*_\omega$. Clearly, the members of C_ω are pairwise disjoint and are either maximal ordinal elements in $O(R)$, or else the union of a transfinite chain of such elements. Furthermore, it is clear that $R_\omega = \bigcup C_\omega$.

Suppose now that $0 < \sigma < \tau$ is any limit ordinal, and suppose the sets C_η have been constructed for all $\eta < \sigma$. We may repeat the argument in the previous paragraph to construct the set C_σ. Using the set C_σ in place of C_0, we can with minor modifications repeat the arguments for the construction of the sets C_n to construct $C_{\sigma+n}$ for any $n < \omega$. In particular, we have the following result:

Theorem 3.3. *If R is a nonempty, preordered root system in which every filet configuration is finite, then there exists an ordinal τ and an ascending chain $C = \{R_\sigma : \sigma < \tau\}$ of lowersets of R such that $R = \bigcup C$, where each $R_\sigma = \bigcup C_\sigma$, and the sets C_σ are constructed as follows:*

1. *The set C_0 consists of all maximal linear elements of $O(R)$;*

2. *If $\sigma < \tau$ has an immediate predecessor, then $X \in C_\sigma$ if and only if $X \in C_{\sigma-1}$ or X is the maximal ordinal extension of the union of two or more members of $C_{\sigma-1}$.*

3. *If $\sigma < \tau$ is a limit ordinal, then $X \in C_\sigma$ if and only if X is the supremum of a transfinite sequence $S = \{Y_\eta : \eta < \sigma\}$, where $Y_\eta \in C_\eta$ for all $\eta < \sigma$ or the maximal ordinal extension of such a sequence, when such exists.*

Theorem 3.3 is a result of applying the classification scheme of Lemma 2.1 to the skeleton of a preordered root system in which every filet configuration is finite. The reader will observe, however, that the decomposition in Theorem 3.3 does not explicitly use the skeleton in its description — it is described only in terms of lowersets. If we forego the use of maximal ordinal extensions, we can place this decomposition in a more general setting. Before doing so, we introduce a bit of terminology.

Two elements are *orthogonal* in a preordered set R provided they have no lower bound in R (other than the least element, if such exists). A subset of R is orthogonal if its elements are pairwise orthogonal.

Definition 3.4. *Let R be a preordered set and let S be a lowerset of R. We say that S admits an ordinal decomposition in $O(R)$ provided there exists an ordinal τ and an ascending chain $C = \{R_\sigma : \sigma < \tau\}$ of lowersets of R such that $S = \bigcup C$, where $R_\sigma = \bigcup C_\sigma$, and the sets C_σ are constructed as follows:*

1. *Each set C_σ is an orthogonal family of lowersets of S;*
2. *If $\sigma < \tau$ is not a limit ordinal, then $x \in C_\sigma$ if and only if $x \in C_{\sigma-1}$, or x is a proper ordinal extension of the union of two or more members of $C_{\sigma-1}$;*
3. *If $0 < \sigma < \tau$ is a limit ordinal, then $x \in C_\sigma$ if and only if x is a (possibly trivial) ordinal extension of a chain $S = \{b_\eta : \eta < \sigma\}$, where $b_\eta \in C_\eta$ for all $\eta < \sigma$.*

We call the sets C_σ the levels of the decomposition.

We know by Theorem 3.3 that if R is a preordered root system in which every filet configuration is finite, then R itself admits an ordinal decomposition whose initial set consists of proper linear elements. The converse is also true.

Theorem 3.5. *If R is a preordered set, then the following are equivalent:*

1. *R is a root system in which every filet configuration is finite;*
2. *R admits an ordinal decomposition whose initial level consists of proper linear elements.*

Proof. We need only prove that Claim (2) implies Claim (1). To see that R is a root system, suppose that j and k are incomparable members of R. There exist least ordinals $\alpha, \beta < \tau$ such that $j \in Z_\alpha$ and $k \in Z_\beta$. We may assume that $\alpha \leq \beta$. Since C_β is an orthogonal set, there exist unique $X, Y \in C_\beta$ such that $j \in X$ and $k \in Y$. It will suffice to prove that $X \neq Y$.

Since C_0 consists of pairwise orthogonal linear elements, if $\beta = 0$, there is nothing to show. Suppose $0 < \beta$. Since $k \in Z_\beta$ and $k \notin Z_\eta$ for $\eta < \beta$, Y must be a proper ordinal extension of the union of a set D constructed in one of two ways: If β is a limit ordinal, then D is an ascending chain such that $D \cap C_\eta \neq \emptyset$ for all $\eta < \beta$; if β is not a limit ordinal, then D is subset of $C_{\beta-1}$ containing at least two elements. In either event, we must have $\bigcup D \subset \downarrow j \subseteq Y$. Thus, Y is an ordinal extension of $\downarrow j$. Since j is therefore comparable to every element of Y, it follows that $X \nsubseteq Y$. The fact that C_β is an orthogonal set implies that $X \neq Y$.

We now know that R is a root system. To complete the proof, we will show that $R/\texttt{Ord}$ satisfies DCC. Let $\texttt{Ord}(x), \texttt{Ord}(y) \in R/\texttt{Ord}$ be such that $\texttt{Ord}(x) \prec \texttt{Ord}(y)$. By assumption, there exist least ordinals $\alpha, \beta < \tau$ such that $\downarrow x \subseteq Z_\alpha$ and $\downarrow y \subseteq Z_\beta$. Consequently, there exist unique $U \in C_\alpha$ and $V \in C_\beta$ such that $x \in U$ and $y \in V$. Since α and β are minimal, both U and V must be ordinal elements of $O(R)$ and as such are either comparable or disjoint by Proposition 3.1. Since $x \leq y$ by assumption, we know that U and V are comparable; indeed, we must have $U \subseteq V$. We cannot have $U = V$ (that is, $\alpha = \beta$) since $\downarrow y$ is not an ordinal extension of $\downarrow x$ by assumption. Since the sets C_σ are all orthogonal sets in $O(R)$ (in fact, they are pairwise disjoint sets), it now follows that $Z_\alpha \subset Z_\beta$. Consequently, any descending chain D in $R/\texttt{Ord}$ induces a descending chain in $\{Z_\sigma : \sigma < \tau\}$ order isomorphic to D. Therefore, $R/\texttt{Ord}$ satisfies DCC. The desired result is now a consequence of Lemma 2.5. □

An ordinal decomposition whose initial level consists of proper linear elements will be called a *linear-based* ordinal decomposition. Theorem 3.5 tells us that a preordered set admits a linear-based ordinal decomposition if and only if it is a root system which satisfies the conditions of Theorem 3.3. In particular, if R admits a linear-based ordinal decomposition, then we may assume that

- R is a root system in which every filet configuration is finite, and
- R admits a linear-based ordinal decomposition in which each level contains either maximal ordinal elements of $O(R)$ or the unions of transfinite sequences of such elements.

We will call the decomposition of Theorem 3.3 a *canonical* linear-based ordinal decomposition. It is possible to devise structural conditions on the root system which afford considerable control over canonical ordinal decompositions. Such conditions are presented in detail for relatively normal lattices in Hart and Tsinakis [6] and can, with a modicum of effort from the reader, be translated into the context of preordered sets. We conclude this section by considering a few such restrictions that might be of interest to theoretical computer scientists.

Corollary 3.6. *If R is a nonempty partially ordered root system satisfying DCC, then the following are equivalent:*

1. *For each $r \in R$, there exists an ordinal $k < \omega$ such that every descending chain in $\downarrow r$ contains at most k elements and one contains exactly k elements;*

2. *There exists an ordinal* $N \leq \omega$ *such that* $R = \bigcup\{B_n : n < N\}$, *where the sets* B_n *are the* n*-levels of the classification scheme of Lemma 2.1.*

Proof. Suppose that Claim (1) is met, and let $r \in R$. It will suffice to show that there exist $n < \omega$ such that $r \in B_n$. Let n_r be the ordinal associated with r by Claim (1), and let C be a descending chain in $\downarrow r$ containing exactly n_r elements. For purposes of notation, let $C = \{c_j : j < n_r\}$, with $c_{j+1} < c_j$ for $j < n_r$. Note that we must have $r = c_0$. Since c_{n_r-1} is the least element of C, it follows that c_{n_r-1} must be minimal in R and hence a member of B_0.

Consider the element c_{n_r-2}. If $c_{n_r-2} \notin B_1$, then c_{n_r-2} is not minimal in $R \setminus B_0$. Let $x \in R \setminus B_0$ be a minimal element below c_{n_r-2}. It follows that x exceeds some $y \in B_0$. Now the chain $r > c_1 > ... > c_{n_r-2} > x > y$ is a descending chain in $\downarrow r$ containing $n_r + 1$ elements — contrary to the choice of n_r. Hence, we must have $c_{n_r-2} \in B_1$.

By repeating the above argument, we can conclude that $c_j \in B_{n_r-j-1}$ for all $j \leq n_r$. In particular, $r \in B_{n_r-1}$.

Conversely, suppose that Claim (2) is met. Let $r \in R$ and let $n_r < \omega$ be the smallest ordinal $n < \omega$ such that $r \in B_n$. Let $C = \{c_j : j \leq N\}$ be a maximal descending chain in $\downarrow r$, where $r = c_0$ and $c_{j+1} < c_j$ for $j < N$. By Claim (2), for each $j < N$, there exists a smallest ordinal n_j such that $c_j \in B_{n_j}$. We know that $c_N \in B_0$ and $c_0 \in B_{n_r}$. It follows that $N \leq n_r + 1$.

To complete the proof, we must find a descending chain in $\downarrow r$ containing exactly $n_r + 1$ elements. If $r \in B_0$, there is nothing to prove; hence, suppose that $0 < n_r$. Since n_r is the smallest ordinal k such that $r \in B_k$, we know that $r \notin B_{n_r-1}$. Hence, there exist $c \in B_{n_r-1}$ such that $c < r$.

Now, if it is true that every $c \in B_{n_r-1}$ which is below r is also a member of B_{n_r-2}, then we would have r minimal in $B \setminus B_{n_r-2}$. This, however, implies that $r \in B_{n_r-1}$ — contrary to assumption. Hence, there must exist some $c_1 < r$ in B_{n_r-1} which is not contained in B_{n_r-2}. It follows that $n_r - 1$ is the smallest ordinal k such that $c_1 \in B_k$. Repeating this argument with c_1 in place of r, we can find a $c_2 \in B_{n_r-2}$ such that $c_2 < c_1 < r$ which is not contained in B_j for any $j < n_r - 2$. Continuing in this fashion, we construct a descending chain $c_{n_r} < c_{n_r-1} < ... < c_2 < c_1 < r$ in $\downarrow r$, where $c_{n_r} \in B_0$. □

The following result can be proven quite easily by mimicking the arguments contained in the proof of Corollary 3.6.

Corollary 3.7. *If* R *is a nonempty partially ordered root system satisfying DCC, then the following are equivalent*

1. *There exists an ordinal* $0 < N < \omega$ *such that every descending chain in* R *contains at most* N *elements, and one contains exactly* N *elements;*

2. *There exists an ordinal* $0 < N < \omega$ *such that* $R = \bigcup\{B_n : n < N\}$*, where the sets* B_n *are as described above.*

4 Information Systems

In this section, we present an application of the theory developed in the previous sections. In particular, we will demonstrate that it is possible to obtain ordinal decompositions of certain information systems in terms of certain information subsystems and characterize those systems which admit such decompositions. We begin with some introductory remarks and definitions.

In the late 1960's, Dana Scott introduced continuous lattices into theoretical computer science as a means of providing models for spaces on which one could define computable functions. However, these objects, while mathematically elegant, were crude in the sense that they contained many elements which could not be assigned a computationally natural meaning. (For more information, see, Gierz, *et al.* [4] and Vickers [14]). In time, the order theoretic models Scott and others considered evolved into what we now call domains — certain classes of algebraic posets. Unfortunately, the level of abstraction required to understand domain theory remained an obstacle to its widespread use. To remedy this problem in some sense, Scott imported from logic the notion of an *information system* to provide a set-theoretic approach to domains. (See Scott [9, 10, 11]).

Viewed from a logician's perspective, an information system for an object or a process is a triple $(S, \mathrm{Con}, \vdash)$, where S is a collection of propositions (or instructions) concerning the object or process, Con is a collection of finite subsets of S which are somehow "consistent" with one another, and $\vdash$ is a relation of interdependence between members of Con. The members of S are thought of as providing simple bits of information about the object or process and are therefore called *tokens*. The set Con is called the *consistency predicate*, and $\vdash$ is known as a relation of *entailment*.

The precise meanings of consistency and entailment are left to the system's designer. However, one can often understand $A \in \mathrm{Con}$ to mean that the tokens in A may be assimilated, understood, executed, *etc.* without encountering contradiction or conflict; and one can often understand $A \vdash B$ to mean that the information content of A "implies" or "refines" that of B.

An information system is assumed to obey certain common sense properties normally associated with the notions of consistency and entailment. These properties are made mathematically precise in the following definition. (In this definition and all the work that follows, we let $\mathtt{Fin}(S)$ denote the set of all finite subsets of a set S, partially ordered by set-inclusion.)

Definition 4.1. *An information system is a triple* $\mathbb{S} = (S, \mathrm{Con}, \vdash)$*, where*

- *IS1* S *is a set;*

- *IS2* Con *is a nonempty lowerset of* $\texttt{Fin}(S)$ *such that* $\bigcup \text{Con} = S$*;*
- *IS3* $\vdash$ *is a preorder on* Con *such that,*
  1. $A \vdash B$ *for all* $B \subseteq A$*;*
  2. *whenever* $A, B, C \in \text{Con}$*,* $A \vdash B$*, and* $A \vdash C$*, then* $B \cup C \in \text{Con}$ *and* $A \vdash (B \cup C)$*.*

We point out that Definition 4.1 is the one used in Hart and Tsinakis [5] and differs slightly from the classical definition of an information system found in Scott [10], and Davey and Priestley [3]. In particular, these authors define entailment as a binary operation on Con $\times$ S rather than on Con $\times$ Con. The difference is semantic only; a quick comparison of axioms will convince the reader that our definition of entailment yields the classical definition, while the classical definition of entailment can be used to build our definition. The primary advantage of our definition is that it allows us to view $(\text{Con}, \vdash)$ as a preordered set.

Let $\mathbb{S} = (S, \text{Con}, \vdash)$ be an information system. A member B of Con is *initial* provided B is entailed by the empty set; that is, provided $B \in \downarrow\emptyset$. Let $\text{Con}^* = \text{Con} \setminus \downarrow\emptyset$ denote the set of all *noninitial* members of Con.

Let **RtSys** denote the class of all information systems $(S, \text{Con}, \vdash)$ for which $(\text{Con}^*, \vdash)$ is a root system. (Note that it is not possible, in general, for $(\text{Con}, \vdash)$ to be a root system since each of its members entails the empty set.) We have the following result as a direct consequence of Theorem 3.5.

Corollary 4.2. *If* $\mathbb{S} = (S, \text{Con}, \vdash)$ *is an information system, then the following are equivalent:*

1. $\mathbb{S}$ *is a member of* ***RtSys****, and every filet configuration in* $(\text{Con}, \vdash)$ *is finite;*
2. $(\text{Con}^*, \vdash)$ *admits an ordinal decomposition in* $O(\text{Con}^*)$ *whose initial level consists of proper linear elements.*

Corollary 4.2 provides a decomposition of $(\text{Con}^*, \vdash)$ in terms of lowersets of $(\text{Con}^*, \vdash)$. We can, however, use this result to obtain an ordinal decomposition of the information system $\mathbb{S}$ in terms of certain subsystems.

Definition 4.3. *Let* $\mathbb{S} = (S, \text{Con}, \vdash)$ *and* $\mathbb{T} = (T, \text{Con}_T, \vdash_T)$ *be information systems. We say that* $\mathbb{T}$ *is a subinformation system of* $\mathbb{S}$ *provided* $T \subseteq S$*,* $\text{Con}_T \subseteq \text{Con}$*, and* $\vdash_T \subseteq \vdash$*. If, in addition,* Con_T *is a lowerset of* $(\text{Con}, \vdash)$*, we will say that* $\mathbb{T}$ *is a full subinformation system of* $\mathbb{S}$*.*

Let $\mathbb{S} = (S, \text{Con}, \vdash)$ and $\mathbb{T} = (T, \text{Con}_T, \vdash_T)$ be information systems. If $\mathbb{T}$ is a subinformation system of $\mathbb{S}$, then we will write $\mathbb{T} \sqsubseteq \mathbb{S}$. Let $\texttt{FSub}(\mathbb{S})$ denote the set of all full subinformation systems of $\mathbb{S}$. It is clear that $\texttt{FSub}(\mathbb{S})$ is a poset under $\sqsubseteq$; however, we can say considerably more than this.

Lemma 4.4. *Let* $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ *be an information system, let* $\mathbb{T} = (T, \mathrm{Con}_T, \vdash_T) \in \mathtt{FSub}(\mathbb{S})$*, and let* $L \in O(\mathrm{Con})$*. The assignments* $\mathbb{T} \mapsto \mathrm{Con}_T$ *and* $L \mapsto (\bigcup L, L, \vdash_L)$ *(where* $\vdash_L$ *is the restriction of* $\vdash$ *to* L*) provide an order isomorphism between* $\mathtt{Fsub}(\mathbb{S})$ *and* $O(\mathrm{Con})$*.*

Lemma 4.4 follows directly from the information system axioms in Definition 4.1. This isomorphism allows us to export the ordinal decomposition of Corollary 4.2 into $\mathtt{FSub}(\mathbb{S})$ thereby giving us an ordinal decomposition of $\mathbb{S}$ in terms of its full subinformation systems. Before we state this result, however, let us examine the structure of $\mathtt{FSub}(\mathbb{S})$.

Let $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ be an information system. In light of Lemma 4.4, we know that $\mathtt{FSub}(\mathbb{S})$ is a bialgebraic (algebraic and dually algebraic), distributive lattice. Joins and meets in this lattice are easily described: Let $\mathbf{F} = \{\mathbb{T}_i : i \in I\}$ be any family of full subinformation systems of $\mathbb{S}$, where each $\mathbb{T}_i = (T_i, \mathrm{Con}_i, \vdash_i)$. Then we know

- $\bigwedge \mathbf{F} = \mathbb{S}$ if $F = \emptyset$,
- $\bigwedge \mathbf{F} = (\bigcap\{T_i : i \in I\}, \bigcap\{\mathrm{Con}_i : i \in I\}, \bigcap\{\vdash_i : i \in I\})$ if $F \neq \emptyset$;
- $\bigvee \mathbf{F} = (\bigcup \downarrow\emptyset, \downarrow\emptyset, \vdash_{\downarrow\emptyset})$ if $F = \emptyset$ (here, $\downarrow\emptyset$ is taken in $(\mathrm{Con}, \vdash)$);
- $\bigvee \mathbf{F} = (\bigcup\{T_i : i \in I\}, \bigcup\{\mathrm{Con}_i : i \in I\}, \bigcup\{\vdash_i : i \in I\})$ if $F \neq \emptyset$.

Furthermore, the compact, join-prime (CJP) members of $\mathtt{FSub}(\mathbb{S})$ are precisely those triples $\mathbb{IS}(A) = (\bigcup \downarrow A, \downarrow A, \vdash_{\downarrow A})$, where $A \in \mathrm{Con}^*$. (For details, the reader is encouraged to consult Hart and Tsinakis [5].)

In Definition 2.2, we introduced the notion of an ordinal extension in a preordered set. It is possible to give a sensible definition of ordinal extensions within an information system as well. The following definition spells out what we mean when we say one information system is an ordinal extension of another.

Definition 4.5. *Let* $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ *be an information system and let* $\mathbb{T} = (T, \mathrm{Con}_T, \vdash_T)$ *be a subinformation system of* $\mathbb{S}$*. We say that* $\mathbb{S}$ *is an ordinal extension of* $\mathbb{T}$ *provided*

- $A \vdash B$ *for all* $B \in \mathrm{Con}_T$ *and* $A \in \mathrm{Con} \setminus \mathrm{Con}_T$*;*
- *the set* $\mathrm{Con} \setminus \mathrm{Con}_T$ *is a pre-ordered chain in* $(\mathrm{Con}, \vdash)$*.*

This definition is really just a modification of Definition 2.2, as the following result demonstrates.

Lemma 4.6. *Let* $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ *be an information system and let* $\mathbb{X} = (X, \mathrm{Con}_X, \vdash_X)$ *and* $\mathbb{Y} = (Y, \mathrm{Con}_Y, \vdash_Y)$ *be full subinformation systems of* $\mathbb{S}$*. If* $\mathbb{X} \sqsubseteq \mathbb{Y}$*, then the following are equivalent:*

1. $\mathbb{Y}$ *is an ordinal extension of* $\mathbb{X}$ *(in the sense of Definition 4.5);*

2. Con_Y *is an ordinal extension of* Con_X *in* $O(\mathrm{Con})$ *(in the sense of Defintion 2.2);*

3. $\mathbb{Y}$ *is an ordinal extension of* $\mathbb{X}$ *in* $\mathtt{FSub}(\mathbb{S})$ *(in the sense of Definition 2.2).*

Proof. The equivalence of Claims 2 and 3 follow at once from Lemma 4.4; we will prove the equivalence of Claims 1 and 2.

If $\mathbb{Y} = \mathbb{X}$, clearly there is nothing to show. Suppose that $\mathbb{Y}$ is a proper ordinal extension of $\mathbb{X}$; that is, assume $\mathrm{Con}_Y \setminus \mathrm{Con}_X \neq \emptyset$. We must prove that in $O(\mathrm{Con})$, $\downarrow\mathrm{Con}_Y \setminus \downarrow\mathrm{Con}_X$ is a chain, each member of which contains Con_X. Let $\mathcal{F} = \downarrow\mathrm{Con}_Y \setminus \downarrow\mathrm{Con}_X$ and let $L \in \mathcal{F}$. It follows that L must contain members of $\mathrm{Con}_Y \setminus \mathrm{Con}_X$; let A be such a member. By Definition 4.5, we know $A \vdash_Y B$ for all $B \in \mathrm{Con}_X$; thus, L contains Con_X.

It remains to prove that $\mathcal{F}$ is a chain in $O(\mathrm{Con})$. Suppose that I, J are incomparable members of $\downarrow\mathrm{Con}_Y$. It follows that there exist $A \in I \setminus (I \cap J)$ and $B \in J \setminus (I \cap J)$. Clearly, A and B are *entailment independent*; that is, $A \not\vdash_Y B$ and $B \not\vdash_Y A$; consequently, by Definition 4.5, both A and B must be members of Con_X. Since no member of $I \cap J$ can be a member of $\mathrm{Con}_Y \setminus \mathrm{Con}_X$ (otherwise we would have $A, B \in I \cap J$ by Definition 4.5), it follows that $I, J \in \downarrow\mathrm{Con}_X$. We now see that $\mathcal{F}$ is a chain in $O(\mathrm{Con})$.

Conversely, suppose that $\downarrow\mathrm{Con}_Y$ is an ordinal extension of $\downarrow\mathrm{Con}_X$ in $\mathcal{L}(\mathrm{Con})$ in the sense of Definition 2.2. Suppose that $A, B \in \mathrm{Con}_Y \setminus \mathrm{Con}_X$. It follows that $\downarrow A = \{C \in \mathrm{Con} : A \vdash_Y C\}$ and $\downarrow B = \{C \in \mathrm{Con} : B \vdash_Y C\}$ are members of $\downarrow\mathrm{Con}_Y \setminus \downarrow\mathrm{Con}_X$ and therefore must be comparable in $O(\mathrm{Con})$ by Definition 2.2. Consequently, either $A \vdash_Y B$ or $B \vdash_Y A$. Furthermore, it follows from Definition 2.2 that $\downarrow A$ must contain Con_X; hence, $A \vdash_Y C$ for all $C \in \mathrm{Con}_X$. Therefore, $\mathbb{Y}$ is an ordinal extension of $\mathbb{X}$. □

The following result is a direct conseqence of Lemma 4.6 and Corollary 4.2.

Corollary 4.7. *For an information system* $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ *the following are equivalent:*

1. $(\mathrm{Con}^*, \vdash)$ *is a root system in which every filet configuration is finite;*

2. *There exists an ordinal* τ *and an ascending chain* $\mathcal{C} = \{\mathbb{R}_\sigma : \sigma < \tau\}$ *of full subinformation systems of* $\mathbb{S}$ *such that* $\mathbb{S} = \bigvee \mathcal{C}$*, and each* $\mathbb{R}_\sigma \in \mathcal{C}$ *is the join in* $\mathtt{FSub}(\mathbb{S})$ *of a set* C_σ *whose members are constructed as follows:*

   (a) C_0 *consists of pairwise disjoint full subinformation systems whose consistency predicates are chains under entailment;*

(b) *if $\sigma < \tau$ has an immediate predecessor, then $\mathbb{X} \in C_\sigma$ if and only if $\mathbb{X} \in C_{\sigma-1}$ or else $\mathbb{X}$ is a proper ordinal extension of the join of two or more members of $C_{\sigma-1}$;*

(c) *If $0 < \sigma < \tau$ is a limit ordinal, then $\mathbb{X} \in C_\sigma$ if and only if there exists an ascending sequence $\{\mathbb{Y}_\eta : \eta < \sigma\}$, where $\mathbb{Y}_\eta \in C_\eta$, such that $\mathbb{X}$ is either the join of this sequence or a proper ordinal extension of this join, if such exists.*

This decomposition theorem implies an intriguing fact about the nature of consistency and entailment in information systems which satisfiy its conditions. Let $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ be an information system and let $\mathbb{T} = (T, \mathrm{Con}_T, \vdash_T)$ be a subinformation system of $\mathbb{S}$. We say that $\mathbb{T}$ is *fully consistent* in $\mathbb{S}$ provided $\mathtt{Fin}(T) \subseteq \mathrm{Con}$; that is, provided every finite subset of T is consistent in $\mathbb{S}$ (though not necessarily in $\mathbb{T}$). If $\mathbb{S}$ is fully consistent in itself, we simply say that $\mathbb{S}$ is fully consistent.

Proposition 4.8. *Let $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ be an information system and let $\mathbb{T} = (T, \mathrm{Con}_T, \vdash_T)$ be a subinformation system of $\mathbb{S}$. If $\mathbb{S}$ is a proper ordinal extension of $\mathbb{T}$, then $\mathbb{S}$ is fully consistent.*

Proof. We first prove that $\mathbb{T}$ is fully consistent in $\mathbb{S}$. Let $F \in \mathtt{Fin}(T)$ and let $x \in F$. Since $\mathbb{S}$ is a proper ordinal extension of $\mathbb{T}$, there exist $A \in \mathrm{Con} \setminus \mathrm{Con}_T$. By Axiom IS1, $\{x\} \in \mathrm{Con}_T$; hence, the first condition of Definition 2.2 implies that $A \vdash \{x\}$. Therefore, Axiom IS4 implies that $F \in \mathrm{Con}$.

Now, suppose that $\{x, y\} \subseteq S \setminus T$. Since $\{x\}, \{y\} \in \mathrm{Con} \setminus \mathrm{Con}_T$ and since $\mathrm{Con} \setminus \mathrm{Con}_T$ is a pre-ordered chain in $(\mathrm{Con}, \vdash)$, either $\{x\} \vdash \{y\}$ or $\{y\} \vdash \{x\}$. Hence, $\{x, y\} \in \mathrm{Con} \setminus \mathrm{Con}_T$ by Axiom IS4. If we now assume $F \in \mathtt{Fin}(S \setminus T)$, a simple induction argument on the cardinality of F proves that $F \in \mathrm{Con} \setminus \mathrm{Con}_T$. □

Proposition 4.8 tells us that, for an information system $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ satisfying Corollary 4.7, consistency and entailment are both completely determined by the ordinal decomposition; and, furthermore, are determined in a simple inductive manner. Indeed, each level of the decomposition consists of a pairwise orthogonal family of fully consistent subsystems. Assume that the decomposition is canonical. To create one level from its predecessor, we partition the predecessor level into disjoint sets; the joins (in $\mathtt{FSub}(\mathbb{S})$) of some of these sets may admit proper ordinal extensions and some may not. If the join of a set does not admit a proper ordinal extension, we bring its members directly into the next level; this action affects neither the consistency predicate nor the entailment relation. Suppose the join of a subset D of the previous level does admit a proper ordinal extension. We bring its maximal ordinal extension (call it $\mathbb{T}$) into the next level. In so doing, we add information about the consistency predicate and the entailment relation:

- We learn that every finite subset of tokens in $\bigvee D$ is consistent.
- Entailment is left unchanged in $\bigvee D$.
- We learn of a new family of consistent subsets of S; namely those appearing in $\mathrm{Con}_T \setminus \mathrm{Con}_{\bigvee D}$. Moreover, we learn that this family forms a chain under entailment.
- We learn that each new consistent subset of S appearing in $\mathrm{Con}_T \setminus \mathrm{Con}_{\bigvee D}$ entails every member of $\mathrm{Con}_{\bigvee D}$.

Furthermore, since the initial level of the decomposition consists of subsystems whose consistency predicates form chains under entailment, this knowledge we gain from the levels is all there is to know about the consistency predicate and the entailment relation of $\mathbb{S}$.

Of course, we may apply the more specialized decomposition theorems presented in Section 3 to information systems as well. We conclude this section by restating these results in the terminology of this section. The proofs are straightforward and left to the reader.

Corollary 4.9. *Let $\mathbb{S} = (S, \mathrm{Con}, \vdash)$ be an information system and let N be a positive integer. The following are equivalent:*

1. *$(\mathrm{Con}^*, \vdash)$ is a root-system, and $(\mathrm{Con}, \vdash)$ has filet-height N;*
2. *$\mathbb{S}$ admits a canonical, linear-based ordinal decomposition in $\texttt{FSub}(\mathbb{S})$ which terminates at N.*

Corollary 4.10. *For an information system $\mathbb{S} = (S, \mathrm{Con}, \vdash)$, and positive integer N the following are equivalent:*

1. *$(\mathrm{Con}^*, \vdash)$ is a root-system containing exactly N distinct maximal preordered chains;*
2. *$(\mathrm{Con}^*, \vdash)$ is a root-system, every entailment-independent subset of* Con *contains at most N members, and one such set contains exactly N members;*
3. *$\mathbb{S}$ admits a canonical finite, linear-based ordinal decomposition in $\texttt{FSub}(\mathbb{S})$ whose initial level contains exactly N members.*

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