ORDERED GROUPS WITH A CONUCLEUS

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Abstract.
Our work proposes a new paradigm for the study of various classes of cancellative residuated lattices by viewing these structures as lattice-ordered groups with a suitable operator (a conucleus). One consequence of our approach is the categorical equivalence between the variety of cancellative commutative residuated lattices and the category of abelian lattice-ordered groups endowed with a conucleus whose image generates the underlying group of the lattice-ordered group. In addition, we extend our methods to obtain a categorical equivalence between ΠMTL-algebras and product algebras with a conucleus. Among the other results of the paper, we single out the introduction of a categorical framework for making precise the view that some of the most interesting algebras arising in algebraic logic are related to lattice-ordered groups. More specifically, we show that these algebras are subobjects and quotients of lattice-ordered groups in a "quantale like" category of algebras.

1. Introduction

In this section, we provide an outline of the contents of the paper. Definitions of concepts not defined here will be given in subsequent sections.

A *residuated lattice-ordered monoid*, or a *residuated lattice* for short, is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice; $\langle L, \cdot, e\rangle$ is a monoid; and for all $x, y, z \in \mathbf{L}$,

$$x \cdot y \leq z \;\Leftrightarrow\; x \leq z/y \;\Leftrightarrow\; y \leq x\backslash z.$$

The elimination of the requirement that a residuated lattice have a least element has led to the development of a surprisingly rich theory that includes the study of various important varieties of cancellative

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residuated lattices, such as the variety of lattice-ordered groups. Refer, for example, to [18], [5], [2], [21] and [8]. These varieties are the focus of the present paper.

Our work initiates a systematic study of the relationship of cancellative varieties of residuated lattices and lattice-ordered groups. In what follows, we will use the term *conucleus* for an interior operator σ on a lattice-ordered group $\mathbf{G}$ that fixes the group identity and whose image is a submonoid of $\mathbf{G}$. The cornerstone of our work is a categorical equivalence between a subclass of cancellative residuated lattices and a category of lattice-ordered groups endowed with a conucleus. More specifically, let $\mathcal{LG}_{cn}$ be the category with objects $\langle \mathbf{G}, \sigma \rangle$, consisting of a lattice-ordered group $\mathbf{G}$ augmented with a conucleus σ such that the underlying group of the lattice-ordered group $\mathbf{G}$ is the group of left quotients of the underlying monoid of $\sigma(\mathbf{G})$. The morphisms of $\mathcal{LG}_{cn}$ are lattice-ordered group homomorphisms that commute with the designated conuclei. Let $\mathcal{ORL}$ be the category each object of which is a cancellative residuated lattice whose underlying monoid is a right reversible monoid. We will refer to these residuated lattices as *Ore residuated lattices.* (Recall that a monoid $\mathbf{M}$ is right reversible if any two principal semigroup ideals of $\mathbf{M}$ have a non-empty intersection: $Ma \cap Mb \neq \emptyset$, for all $a, b \in M$.) The morphisms in $\mathcal{ORL}$ are residuated lattice homomorphisms. Then the categories $\mathcal{LG}_{cn}$ and $\mathcal{ORL}$ are equivalent. By prescribing special properties for the conucleus or by restricting the class of objects, we obtain restricted categorical equivalences between subcategories of $\mathcal{LG}_{cn}$ and subcategories of $\mathcal{ORL}$. For example, if $\mathcal{CLG}_{cn}$ is the full subcategory of $\mathcal{LG}_{cn}$ consisting of objects whose first components are abelian lattice-ordered groups, and if $\mathcal{CCan RL}$ is the variety of commutative cancellative residuated lattices, then $\mathcal{CLG}_{cn}$ and $\mathcal{CCan RL}$ are equivalent.

To further illuminate the equivalence discussed above, we consider the category, $\mathcal{RL}^{\times}$, whose objects are residuated lattices and whose morphisms are monoid homomorphisms that are also residuated maps. Then it will be shown that the objects of $\mathcal{ORL}$ are subobjects of lattice-ordered groups in the category $\mathcal{RL}^{\times}$. In particular, the members of $\mathcal{CCan RL}$ encompass all the subobjects of abelian lattice-ordered groups in the category $\mathcal{RL}^{\times}$. This perspective also sheds new light into the main results in [24], [10] and [12].

Indeed, a fundamental result in the theory of MV-algebras, due to Mundici [24], is the categorical equivalence between the category of MV-algebras and the category of unital abelian lattice-ordered groups, that is, abelian lattice-ordered groups with a designated strong order unit. Dvurečenskij generalized, in [10], the Mundici correspondence to bounded GMV-algebras and arbitrary unital lattice-ordered groups. Dvurečenskij's result is subsumed by the following result in [12]. Let $\mathcal{IGMV}$ be the variety of integral GMV-algebras and let $\mathcal{LG}^-_{ncl}$ be the category with objects $\langle \mathbf{B}, \gamma \rangle$ consisting of the negative cone, $\mathbf{B}$, of a lattice-ordered group augmented with a nucleus γ on it whose image generates $\mathbf{B}$ as a monoid. Let the morphisms of these categories be algebra homomorphisms. Then the categories $\mathcal{GMV}$ and $\mathcal{LG}^-_{ncl}$ are equivalent.

It will be shown that the last equivalence allows us to view integral GMV algebras as the epimorphic images, in $\mathcal{RL}^\times$, of negative cones of lattice-ordered groups. MV-algebras and bounded GMV-algebras are special epimorphic images of negative cones of abelian lattice-ordered groups and arbitrary lattice-ordered groups, respectively. Hence, some of the most interesting algebras arising in algebraic logic are either subobjects of lattice-ordered groups or epimorphic images of negative cones of lattice-ordered groups in $\mathcal{RL}^\times$.

Motivated by the preceding facts, we ask whether the results of the previous sections can be extended to residuated lattices that are not cancellative. In this setting, an appropriate substitute for the concept of a lattice-ordered group is that of an involutive residuated lattice. By employing an embedding result in [26], we show that every residuated lattice with top element is a subobject, in $\mathcal{RL}^\times$, of an involutive residuated lattice. It's an open question at this time as to whether this correspondence extends to a categorical equivalence.

In the last section of the paper we investigate an application to many-valued logic. More precisely, we establish a categorical equivalence between ΠMTL-algebras and product algebras (i.e., divisible ΠMTL-algebras) with a conucleus which is also a lattice endomorphism and whose image generates the whole algebra. We show, in particular, that for any ΠMTL-algebra $\mathbf{A}$ there exists a unique – up to isomorphism – product algebra $\mathbf{A}^*$ such that $A \subseteq A^*$, $\mathbf{A}$ is closed with respect to the monoid and lattice operations of $\mathbf{A}^*$ and, relative to the implication $\to^*$ in $\mathbf{A}^*$, every element $x \in A^*$ can be written as $x = a \to^* b$, for some elements $a, b \in A$.

2. Basic Facts

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. A map $f : \mathbf{P} \to \mathbf{Q}$ is said to be *residuated* provided there exists a map $f_\star : \mathbf{Q} \to \mathbf{P}$ such that

$$f(x) \leq y \iff x \leq f_\star(y),$$

for all $x \in P$ and $y \in Q$. We refer to $f_\star$ as the *residual* of f. We note that f preserves any existing joins and $f_\star$ preserves any existing meets.

This definition extends to binary maps as follows: Let $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ be posets. A binary map $\cdot : \mathbf{P} \times \mathbf{Q} \to \mathbf{R}$ is said to be *biresiduated* provided there exist binary maps $\backslash : \mathbf{P} \times \mathbf{R} \to \mathbf{Q}$ and $/ : \mathbf{R} \times \mathbf{Q} \to \mathbf{P}$ such that

$$xy \leq z \iff x \leq z/y \iff y \leq x\backslash z,$$

for all $x \in P, y \in Q, z \in R$.

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$. In the event $x\backslash y = y/x$, we write $x \to y$ for the common value. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a “mirror image” obtained by reading terms backwards (i.e., replacing xy by yx and interchanging x/y with $y\backslash x$).

We are interested in the situation where $\cdot$ is a monoid operation with unit element e. In this case, we add the monoid unit to the similarity type and refer to the resulting structure $\mathbf{A} = \langle A, \cdot, \backslash, /, e, \leq \rangle$ as a *residuated partially ordered monoid*. If the partial order is a lattice order, we obtain a purely algebraic structure $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$ called a *residuated lattice-ordered monoid* or a *residuated lattice* for short.

Residuated lattices form a finitely based variety (see, for example, [5] and [21]), denoted by $\mathcal{RL}$.

Given a residuated lattice $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, e \rangle$, an element $a \in A$ is said to be *integral* if $e/a = e = a\backslash e$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral. We denote by $\mathcal{IRL}$ the variety of all integral residuated lattices. Important classes of residuated lattices arise as negative cones of non-integral residuated lattices. The *negative cone* of a residuated lattice $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e \rangle$ is the algebra $\mathbf{L}^- = \langle L^-, \wedge, \vee, \cdot, \backslash_{\mathbf{L}^-}, /_{\mathbf{L}^-}, e \rangle$, where $L^- = \{x \in L \mid x \leq e\}$, $x\backslash_{\mathbf{L}^-} y = x\backslash y \wedge e$ and $x/_{\mathbf{L}^-} y = x/y \wedge e$. It is easy to verify that $\mathbf{L}^-$ is indeed a residuated lattice.

An element $a \in A$ is said to be *invertible* if $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The structures in which every element is invertible are therefore precisely the lattice-ordered groups and the partially ordered groups. Perhaps a word of caution is appropriate here. A lattice-ordered group is usually defined in the literature as an algebra $\mathbf{G} = \langle G, \wedge, \vee, \cdot, ^{-1}, e\rangle$ such that $\langle G, \wedge, \vee\rangle$ is a lattice, $\langle G, \cdot, ^{-1}, e\rangle$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of lattice-ordered groups is term equivalent to the subvariety of $\mathcal{RL}$ defined by the equations $(e/x)x \approx e \approx x(x\backslash e)$; the term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}$, $x\backslash y = x^{-1}y$. We denote by $\mathcal{LG}$ the aforementioned subvariety and refer to its members as lattice-ordered groups, but we will freely use the traditional signature in our computations.

Cancellative residuated lattices are the focus of this paper and are natural generalizations of lattice-ordered groups. Although cancellative monoids are defined by quasi-equations, the class $\mathcal{C}an\mathcal{RL}$ of cancellative residuated lattices is a variety, as the following result demonstrates.

Lemma 2.1. ([2]) *A residuated lattice is cancellative as a monoid if and only if it satisfies the identities $xy/y \approx x \approx y\backslash yx$.* □

The variety of cancellative residuated lattices will be denoted by $\mathcal{C}an\mathcal{RL}$ and that of commutative cancellative residuated lattices by $\mathcal{CC}an\mathcal{RL}$.

As was noted above, a monoid $\mathbf{M}$ is right reversible if any two principal semigroup ideals of $\mathbf{M}$ have a non-empty intersection: $Ma \cap Mb \neq \emptyset$, for all $a, b \in M$. By a result due to Ore (refer to Section 1.10 of [7]), right reversibility, combined with cancellativity, is a sufficient condition for the embeddability of a monoid into a group. Moreover, it is also a necessary condition if the embedding into a group is of the following simple type. We say that a group $\mathbf{G}$ is a group of left-quotients of a monoid $\mathbf{M}$, if $\mathbf{M}$ is a submonoid of $\mathbf{G}$ and every element of G can be expressed in the form $a^{-1}b$ for some $a, b \in M$.

Lemma 2.2.

(1) *A cancellative monoid has a group of left quotients if and only if it is right reversible.*
(2) *A right reversible monoid uniquely determines its group of left quotients. More specifically, let $\mathbf{M}$ be a right reversible monoid and let $\mathbf{G}_1(\mathbf{M})$ and $\mathbf{G}_2(\mathbf{M})$ be groups of left quotients of $\mathbf{M}$. Then there exists a group isomorphism between $\mathbf{G}_1(\mathbf{M})$ and $\mathbf{G}_2(\mathbf{M})$ that fixes the elements of $\mathbf{M}$.* □

A proof of the previous result, due to Dubreil [9], can be found in Section 1.10 of [7].

3. Conuclei and Interior Extractions

An *interior operator* on a poset $\mathbf{P}$ is a map $\sigma : \mathbf{P} \to \mathbf{P}$ with the usual properties of preserving the order, being contracting ($\sigma(x) \leq x$), and being idempotent. Its image, P_σ, satisfies

(3.1) $\max\{a \in P_\sigma : a \leq x\}$ exists for all $x \in P$.

Thus, σ is completely determined by its image by virtue of the formula

(3.2) $\sigma(x) = \max\{a \in P_\sigma : a \leq x\}$.

It follows that there exists a bijective correspondence between all interior operators σ on a poset $\mathbf{P}$ and all subposets $\mathbf{O}$ of $\mathbf{P}$ satisfying the condition

(3.3) $\max\{a \in O : a \leq x\}$ exists for all $x \in P$.

We note, for future reference, that if a subposet $\mathbf{O}$ of a poset $\mathbf{P}$ satisfies (3.3), then it is closed under any existing joins in $\mathbf{P}$. That is, if $(x_i : i \in I)$ is an arbitrary family of elements of $\mathbf{O}$ such that ${}^{\mathbf{P}}\bigvee_{i\in I} x_i$ exists, then ${}^{\mathbf{O}}\bigvee_{i\in I} x_i$ exists and ${}^{\mathbf{P}}\bigvee_{i\in I} x_i = {}^{\mathbf{O}}\bigvee_{i\in I} x_i$.

An interior operator σ on a residuated partially ordered monoid $\mathbf{P}$ is said to be a *conucleus* if $\sigma(e) = e$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$, for all $x, y \in P$. The latter condition is clearly equivalent to $\sigma(\sigma(x)\sigma(y)) = \sigma(x)\sigma(y)$, for all $x, y \in P$. In what follows, we will often refer to the elements of P_σ as the *open* elements of $\mathbf{P}$ (relative to σ). An *interior extraction* of a residuated partially ordered monoid $\mathbf{P}$ is a subposet and a submonoid, $\mathbf{Q}$, of $\mathbf{P}$ that satisfies condition (3.3) above. It is clear that if σ is a conucleus on $\mathbf{P}$, then $\mathbf{P}_\sigma$ is an interior extraction of $\mathbf{P}$. Conversely, if $\mathbf{Q}$ is an interior extraction of $\mathbf{P}$, then $\sigma_{\mathbf{Q}} : \mathbf{P} \to \mathbf{P}$ – defined by $\sigma_{\mathbf{Q}}(x) = \max\{a \in Q : a \leq x\}$, for all $x \in P$ – is a conucleus on $\mathbf{P}$. Moreover, this correspondence is bijective.

The next result shows that every interior extraction of a residuated lattice is a residuated lattice on its own right.

Lemma 3.1. *If* $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ *is a residuated lattice and* σ *a conucleus on it, then the algebra* $\mathbf{L}_\sigma = \langle L_\sigma, \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, e\rangle$ *is a residuated lattice – where* $x \wedge_\sigma y = \sigma(x \wedge y)$, $x/_\sigma y = \sigma(x/y)$ *and* $x\backslash_\sigma y = \sigma(x\backslash y)$, *for all* $x, y \in L_\sigma$.

Proof. In view of the preceding discussion, $\mathbf{L}_\sigma$ is a submonoid and a join-subsemilattice of $\mathbf{L}$. It is obviously closed under $\backslash_\sigma$ and $/_\sigma$, and $\wedge_\sigma$ is clearly the meet operation on $\mathbf{L}_\sigma$. We complete the proof by showing that multiplication in $\mathbf{L}_\sigma$ is residuated with residuals $\backslash_\sigma$, and $/_\sigma$. Indeed, for all $x, y, z \in L_\sigma$, $x \leq z/_\sigma y$ is equivalent to $x \leq \sigma(z/y)$, which in turn is equivalent to $x \leq z/y$, since σ is contracting and $x = \sigma(x)$. $\square$

A concept dual to the concept of an interior operator is that of a closure operator. A *closure operator* on a poset $\mathbf{P}$ is a map $\gamma : \mathbf{P} \to \mathbf{P}$ that is order preserving, extensive ($x \leq \gamma(x)$), and idempotent. Its image, P_γ, satisfies

(3.4) $\min\{a \in P_\gamma : x \leq a\}$ exists for all $x \in P$.

Thus, γ is determined by its image via the formula

(3.5) $\gamma(x) = \min\{a \in P_\gamma : x \leq a\}$.

Hence there exists a bijective correspondence between all closure operators γ on a poset $\mathbf{P}$ and all subposets $\mathbf{C}$ of $\mathbf{P}$ satisfying the condition

(3.6) $\min\{a \in C : x \leq a\}$ exists for all $x \in P$.

As in the dual situation, if a subposet $\mathbf{C}$ of a poset $\mathbf{P}$ satisfies (3.6), then it is closed under any existing meets in $\mathbf{P}$.

A closure operator γ on a residuated partially ordered monoid $\mathbf{P}$ is said to be a *nucleus* if $\gamma(x)\gamma(y) \leq \gamma(xy)$, for all $x, y \in P$. In what follows, we will have the occasion to refer to the elements of P_γ as the *closed* elements of $\mathbf{P}$ (relative to γ). A *closure retraction* of a residuated partially ordered monoid $\mathbf{P}$ is a subposet $\mathbf{Q}$, of $\mathbf{P}$ that satisfies condition (3.6) above, and, moreover, for all $x \in P$ and $y \in Q$, $x\backslash y \in Q$ and $y/x \in Q$. If γ is a nucleus on $\mathbf{P}$, then $\mathbf{P}_\gamma$ is a closure retraction of $\mathbf{P}$. Conversely, if $\mathbf{Q}$ is a closure retraction of $\mathbf{P}$, then $\gamma_{\mathbf{Q}} : \mathbf{P} \to \mathbf{P}$ – defined by $\gamma_{\mathbf{Q}}(x) = \min\{a \in Q : x \leq a\}$, for all $x \in P$ – is a nucleus on $\mathbf{P}$. Moreover, this correspondence is bijective. (Refer to [12] for details.)

The next result shows that every closure retraction of a residuated lattice is a residuated lattice on its own right. Its simple proof can be found in [12]

Lemma 3.2. *Let* $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ *be a residuated lattice,* γ *be a nucleus on* $\mathbf{L}$ *and* L_γ *be the closure retraction associated with* γ*. Then the algebraic system* $\mathbf{L}_\gamma = \langle L_\gamma, \wedge, \vee_\gamma, \circ_\gamma, \backslash, /, \gamma(e)\rangle$ *– where* $x \circ_\gamma y = \gamma(x \cdot y)$ *and* $x \vee_\gamma y = \gamma(x \vee y)$ *– is a residuated lattice.* $\square$

4. The Categorical Equivalence

The main result of this section establishes that the categories $\mathcal{LG}_{cn}$ and $\mathcal{ORL}$ are equivalent. Recall that $\mathcal{ORL}$ be the category of Ore residuated lattices and residuated lattice homomorphisms. $\mathcal{LG}_{cn}$ is the category with objects $\langle \mathbf{G}, \sigma \rangle$, consisting of a lattice-ordered group $\mathbf{G}$ augmented with a conucleus σ such that the underlying group of the lattice-ordered group $\mathbf{G}$ is the group of left quotients of the underlying monoid of $\sigma(\mathbf{G})$. The morphisms of $\mathcal{LG}_{cn}$ are lattice-ordered group homomorphisms that commute with the designated conuclei.

We hasten to add that the class $\mathcal{ORL}$ is a proper subclass of the variety of cancellative residuated lattices. For example, it is shown in [2] that the free monoid in any number of generators can serve as the underlying monoid of a residuated lattice. Such a residuated lattice is not Ore, since the free monoid in two or more generators is clearly not right reversible. However, $\mathcal{ORL}$ contains important subvarieties of $\mathcal{RL}$, including the variety of commutative, cancellative residuated lattices. Refer to Section 5 for additional examples of subvarieties of $\mathcal{ORL}$.

Before we establish the promised categorical equivalence we will prove a series of results.

Let $\mathbf{L}$ be an Ore residuated lattice and let $\mathbf{G}(\mathbf{L})$ be the group of left quotients of the underlying monoid of $\mathbf{L}$ (see Lemma 2.2). Lemma 4.2 below shows that there exists a lattice order on $G(L)$ that extends the order of $\mathbf{L}$ and with respect to which $\mathbf{G}(\mathbf{L})$ becomes a lattice-ordered group.

Lemma 4.1. *Let $a^{-1}b, c^{-1}d$ be two typical elements of $\mathbf{G}(\mathbf{L})$, with $a, b, c, d \in L$. Then $a^{-1}b = c^{-1}d$ in $\mathbf{G}(\mathbf{L})$ if and only if there exist $x, y \in L$ such that $xb = yd$ and $xa = yc$.*

Proof. By the definition of $\mathbf{G}(\mathbf{L})$, there exist elements $x, y \in L$ such that $ca^{-1} = y^{-1}x$. Thus, $a^{-1}b = c^{-1}d$ yields successively $ca^{-1}b = d$, $y^{-1}xb = d$ and $xb = yd$. Also $ca^{-1} = y^{-1}x$ implies $xa = yc$. Conversely, if $xa = yc$ and $xb = yd$, then $a^{-1}b = (xa)^{-1}(xb) = (yc)^{-1}yd = c^{-1}d$. □

Retaining the preceding notation, let $\leq$ denote the lattice order of $\mathbf{L}$ and let $\preceq$ denote the binary relation on $G(L)$ defined, for all $a, b, c, d \in L$, by

(4.1) $a^{-1}b \preceq c^{-1}d$ iff there exist $x, y \in L$ such that $xb \leq yd$ and $xa = yc$.

Lemma 4.2. *Let $\mathbf{L}$ be an Ore residuated lattice, let $\mathbf{G}(\mathbf{L})$ be the group of left quotients of the underlying monoid of $\mathbf{L}$, and let $\leq$ and $\preceq$ be defined as above.*

(i) *The binary relation $\preceq$ is the unique lattice order on $G(L)$ that extends $\leq$ and with respect to which $\mathbf{G}(\mathbf{L})$ is a lattice-ordered group.*

(ii) *Finite joins in $\mathbf{L}$ coincide with the corresponding joins in $\mathbf{G}(\mathbf{L})$.*

(iii) *Let $a^{-1}b, c^{-1}d$ be two representative elements of $\mathbf{G}(\mathbf{L})$, with $a, b, c, d \in L$. The join of $a^{-1}b$ and $c^{-1}d$ in $\mathbf{G}(\mathbf{L})$ is given by the formula,*

$$(a^{-1}b) \vee (c^{-1}d) = (xa)^{-1}(xb \vee yd),$$

where x, y are any two elements of L such that $xa = yc$.

Proof. To establish (i), we first determine the positive cone of $\preceq$. Let S be the subset of $G(L)$ defined by

$$S = \{a^{-1}b : a, b \in L, b \geq a\}$$

We claim that S satisfies the following three conditions:

(a) $S \cap S^{-1} = \{e\}$;
(b) $SS \subseteq S$; and
(c) $xSx^{-1} \subseteq S$, for all $x \in G(L)$.

In other words, S is a normal subsemigroup of $\mathbf{G}(\mathbf{L})$ that contains e, but no other elements and its inverse.

It is clear that S satisfies condition (a). To prove condition (b), suppose $a^{-1}b, c^{-1}d \in S$. Let $x, y \in L$ such that $x^{-1}y = bc^{-1}$, that is, $yc = xb$. Then, $(a^{-1}b)(c^{-1}d) = a^{-1}x^{-1}yd = (xa)^{-1}(yd)$. By assumption, $b \geq a$ and $d \geq c$. Thus, $yd \geq yc = xb \geq xa$. It follows that $(xa)^{-1}(yd) = (a^{-1}b)(c^{-1}d) \in S$. This completes the proof of (b).

We next establish (c). Let first $a^{-1}b \in S$ and $c \in L$. Then it is readily seen that $c^{-1}a^{-1}bc \in S$. The proof of $ca^{-1}bc^{-1} \in S$ requires more work. Let $x, y, z, w \in L$ such that $ca^{-1} = x^{-1}y$ and $ybc^{-1} = z^{-1}w$. These equalities can be written alternatively as $xc = ya$ and $wc = zyb$. Now, $ca^{-1}bc^{-1} = x^{-1}ybc^{-1} = x^{-1}z^{-1}w$. Thus, to establish that $ca^{-1}bc^{-1} \in S$, it will suffice to prove that $w \geq zx$. We have $wc = zyb \geq zya = zxc$ – since $b \geq a$, by assumption – and hence $w \geq zx$, by cancellativity. To summarize, we have shown that S is closed under conjugation by c and c^{-1}, for all $c \in L$. Consequently, S is a normal subsemigroup of $\mathbf{G}(\mathbf{L})$, as was to be shown.

As is well known (see, for example, [11], page 13), any subset of a group satisfying conditions (a), (b) and (c), is the positive cone of a partial order on the group in question. In this particular case, the partial order on $G(L)$ with positive cone S is defined by $x \preceq_1 y$ if and only if $x^{-1}y \in S$, for all $x, y \in G(L)$. It is readily seen that $\preceq_1$ is none other than $\preceq$. We also note that (4.1) ensures that any compatible partial order on $\mathbf{G}(\mathbf{L})$ must coincide with $\preceq$.

So far we have shown that $\mathbf{G}(\mathbf{L})$ is a partially ordered group with respect to $\preceq$. Further, it is clear that $\preceq$ extends $\leq$. To complete the proof of (i), we must show that $\preceq$ is a lattice order. For that, we first establish condition (ii) in the statement of the theorem. Denoting the join operations in $\mathbf{L}$ and $\mathbf{G}(\mathbf{L})$ by $\vee^{\mathbf{L}}$ and $\vee^{\mathbf{G}}$, respectively, we need to show that $a \vee^{\mathbf{G}} b = a \vee^{\mathbf{L}} b$, for all $a, b \in L$. Obviously, $a \vee^{\mathbf{L}} b$ is an upper bound of a and b in $\mathbf{G}(\mathbf{L})$. If $c^{-1}d$ is another upper bound of a and b, with $c, d \in L$, then $ca \leq d$ and $cb \leq d$. Thus, $ca \vee^{\mathbf{L}} cb = c(a \vee^{\mathbf{L}} b) \leq d$. This yields, $a \vee^{\mathbf{L}} b \leq c^{-1}d$ and establishes condition (ii).

We next complete the proof of (i) by verifying that $\preceq$ is a lattice order. It is well known and easy to prove – see for example [11], page 67 – that a partially ordered group $\mathbf{G}$ is a lattice-ordered group if and only if, for every $x \in G$, the join $x \vee e$ exists. Specializing in $\mathbf{G}(\mathbf{L})$, let $a, b \in L$. We need to prove that $a^{-1}b \vee^{\mathbf{G}} e$ exists. We have already seen that $b \vee^{\mathbf{G}} a$ exists. Now the map $f_{a^{-1}} : G(L) \to G(L)$, defined by $f_{a^{-1}}(x) = a^{-1}x$, for all $x \in G(L)$, is an order automorphism of $\langle G(L), \preceq\rangle$ and hence it preserves all existing joins. Thus, $f_{a^{-1}}(b \vee^{\mathbf{G}} a) = f_{a^{-1}}(b) \vee^{\mathbf{G}} f_{a^{-1}}(a) = a^{-1}b \vee^{\mathbf{G}} e$ exists.

It remains to prove (iii). Throughout the remainder of the paper we will denote the join operation in $\mathbf{G}(\mathbf{L})$ by $\vee$. Let $a^{-1}b, c^{-1}d$ be two representative elements of $\mathbf{G}(\mathbf{L})$, with $a, b, c, d \in L$. Let $x, y \in L$ be any elements such that $x^{-1}y = ac^{-1}$, that is, $xa = yc$. Such elements exist, since the underlying monoid of $\mathbf{L}$ is right reversible. Then, using the fact that multiplication distributes over joins, we get $(a^{-1}b) \vee (c^{-1}d) = a^{-1}(b \vee ac^{-1}d) = a^{-1}(b \vee x^{-1}yd) = (xa)^{-1}(xb \vee yd)$.

□

As was noted above, the join operation of $\mathbf{G}(\mathbf{L})$ will be denoted by $\vee$. Further, we will use $\leq$ for $\preceq$ and the partial order of $\mathbf{L}$.

Lemma 4.3. *An Ore residuated lattice determines uniquely its lattice-ordered group of left quotients. More specifically, let $\mathbf{L}$ be an Ore residuated lattice and let $\mathbf{G}_1(\mathbf{L})$ and $\mathbf{G}_2(\mathbf{L})$ be lattice-ordered groups of left quotients of $\mathbf{L}$. Then there exists a lattice-ordered group isomorphism between $\mathbf{G}_1(\mathbf{L})$ and $\mathbf{G}_2(\mathbf{L})$ that fixes the elements of $\mathbf{L}$.*

Proof. Let $\leq_1$ and $\leq_2$ denote the lattice-orders of $\mathbf{G}_1(\mathbf{L})$ and $\mathbf{G}_2(\mathbf{L})$, respectively, and let $\cdot_1$ and $\cdot_2$ be the corresponding multiplications. We will use the same symbol $^{-1}$ for the inverse operation in both algebras. In light of Lemma 2.2, there exists a group isomorphism $\varphi : \mathbf{G}_1(\mathbf{L}) \longrightarrow \mathbf{G}_2(\mathbf{L})$ that fixes the elements of $\mathbf{L}$. Let $a^{-1} \cdot_1 b$, $c^{-1} \cdot_1 d$

be two representative elements of $\mathbf{G}_1(\mathbf{L})$, with a, b, c, d in L. Then, by (4.1), $a^{-1} \cdot_1 b \leq_1 c^{-1} \cdot_1 d$ if and only if there exist $x, y \in L$ such that $xb \leq yd$ and $xa = yc$ in $\mathbf{L}$. Thus, again by (4.1), this is equivalent to $a^{-1} \cdot_2 b \leq_2 c^{-1} \cdot_2 d$, that is, $\varphi(a^{-1} \cdot_1 b) \leq_2 \varphi(c^{-1} \cdot_1 d)$. It follows that φ is an order-isomorphism, and hence a lattice-ordered group isomorphism. □

Let $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ be an Ore residuated lattice and let $\mathbf{G}(\mathbf{L})$ be its lattice-ordered group of left quotients. Define $\sigma_{\mathbf{L}} : G(L) \to G(L)$ by

(4.2) $\sigma_{\mathbf{L}}(a^{-1}b) = a\backslash b$, for all $a, b \in L$.

Lemma 4.4. *Let* $\mathbf{L}$*,* $\mathbf{G}(\mathbf{L})$ *and* $\sigma_{\mathbf{L}}$ *be defined as above.*

(i) $\langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}}\rangle$ *is an object in* $\mathcal{LG}_{cn}$*.*
(ii) $\mathbf{L} = \mathbf{G}(\mathbf{L})_{\sigma_{\mathbf{L}}}$*, as residuated lattices.*

Proof. Note first that $\sigma_{\mathbf{L}}$ is well defined. Indeed, let $a^{-1}b, c^{-1}d$ be two elements of $\mathbf{G}(\mathbf{L})$, with $a, b, c, d \in L$, such that $a^{-1}b = c^{-1}d$. In light of Lemma 4.1, there exist elements $x, y \in L$ such that $xb = yd$ and $xa = yc$. Hence, invoking the fact that $\mathbf{L}$ is a cancellative residuated lattice, we get $a\backslash b = xa\backslash xb = yc\backslash yd = c\backslash d$, that is, $\sigma_{\mathbf{L}}(a^{-1}b) = \sigma_{\mathbf{L}}(c^{-1}d)$.

Now, by definition, $a\backslash b$ is the greatest element $z \in L$ such that $az \leq b$ holds in $\mathbf{L}$. But $az \leq b$ holds in $\mathbf{L}$ if and only if $z \leq a^{-1}b$ holds in $\mathbf{G}(\mathbf{L})$. Thus, $a\backslash b = \max\{z : z \in L, z \leq a^{-1}b\}$. It follows (refer to Section 3) that $\mathbf{L}$ is an interior extraction of $\mathbf{G}(\mathbf{L})$ and the associated interior operator $\sigma_{\mathbf{L}}$ is a conucleus.

□

Lemma 4.5. *For every morphism* $\chi : \mathbf{L} \to \mathbf{K}$ *of the category* $\mathcal{ORL}$*, let* $\Omega(\chi) : \langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}}\rangle \to \langle \mathbf{G}(\mathbf{K}), \sigma_{\mathbf{K}}\rangle$ *be defined, for all* $a, b \in L$*, by* $\Omega(\chi)(a^{-1}b) = (\chi(a))^{-1}\chi(b)$*. Then* $\Omega(\chi)$ *is the unique* $\mathcal{LG}_{cn}$*-morphism from* $\langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}}\rangle$ *to* $\langle \mathbf{G}(\mathbf{K}), \sigma_{\mathbf{K}}\rangle$ *extending* χ*.*

Proof. Note first that $\Omega(\chi)$ is well defined. Indeed, suppose that a, b, c, d are elements in L such that $a^{-1}b = c^{-1}d$ in $\mathbf{G}(\mathbf{L})$. Then, by Lemma 4.1, there exist elements $x, y \in L$ such that $xb = yd$ and $xa = yc$. It follows that $\chi(x)\chi(b) = \chi(y)\chi(d)$ and $\chi(x)\chi(a) = \chi(y)\chi(c)$, since $\chi : \mathbf{L} \to \mathbf{K}$ is a homomorphism. Thus, again by Lemma 4.1, $\chi(a)^{-1}\chi(b) = \chi(c)^{-1}\chi(d)$, that is, $\Omega(\chi)(a^{-1}b) = \Omega(\chi)(c^{-1}d)$.

Next, note that any $\mathcal{LG}_{cn}$-morphism from $\langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}}\rangle$ to $\langle \mathbf{G}(\mathbf{K}), \sigma_{\mathbf{K}}\rangle$ that extends χ must be equal to $\Omega(\chi)$. Thus, it will suffice to prove that $\Omega(\chi)$ is a $\mathcal{LG}_{cn}$-morphism. We first show that it is a lattice-ordered group homomorphism. $\Omega(\chi)$ clearly preserves the group operations.

Also, note that the meet operation in $\mathbf{G}(\mathbf{L})$ satisfies $u \wedge v = (u^{-1} \vee v^{-1})^{-1}$, for all $u, v \in G(L)$. Thus, it will suffice to show that $\Omega(\chi)$ preserves finite joins. Let $a^{-1}b, c^{-1}d$ be two representative elements of $\mathbf{G}(\mathbf{L})$, with $a, b, c, d \in L$. In light of Lemma 4.2, the join of $a^{-1}b$ and $c^{-1}d$ in $\mathbf{G}(\mathbf{L})$ is given by the formula, $(a^{-1}b) \vee (c^{-1}d) = (xa)^{-1}(xb \vee yd)$, where x, y are any two elements of L such that $xa = yc$. Now, since χ is a homomorphism and $\Omega(\chi)$ preserves the group operations, we get that $\Omega(\chi)((a^{-1}b) \vee (c^{-1}d)) = (\chi(x)\chi(a))^{-1}(\chi(x)\chi(b) \vee \chi(y)\chi(d))$. Thus, again by Lemma 4.2, $\Omega(\chi)((a^{-1}b) \vee (c^{-1}d)) = (\chi(a)^{-1}\chi(b)) \vee (\chi(c)^{-1}\chi(d)) = \Omega(\chi)(a^{-1}b) \vee \Omega(\chi)(c^{-1}d)$.

Lastly, we need to prove that $\Omega(\chi)$ commutes with the conuclei. Let $a, b \in L$. Then $\Omega(\chi)\sigma_{\mathbf{L}}(a^{-1}b) = \Omega(\chi)(a \backslash b) = \chi(a \backslash b) = \chi(a) \backslash \chi(b) = \sigma_{\mathbf{K}}(\chi(a)^{-1}\chi(b)) = \sigma_{\mathbf{K}}\Omega(\chi)(a^{-1}b)$. Thus, $\Omega(\chi)\sigma_{\mathbf{L}} = \sigma_{\mathbf{K}}\Omega(\chi)$. □

The promised equivalence between the categories $\mathcal{ORL}$ and $\mathcal{LG}_{cn}$ will be witnessed by the following pair of functors $\Omega : \mathcal{ORL} \to \mathcal{LG}_{cn}$ and $\Omega^{-1} : \mathcal{LG}_{cn} \to \mathcal{ORL}$.

Definition 4.6.

(a) *For every object* $\mathbf{L}$ *in* $\mathcal{ORL}$*, let* $\Omega(\mathbf{L}) = \langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}} \rangle$.

(b) *For every morphism* $\chi : \mathbf{L} \to \mathbf{K}$ *of the category* $\mathcal{ORL}$*, let* $\Omega(\chi) : \langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}} \rangle \to \langle \mathbf{G}(\mathbf{K}), \sigma_{\mathbf{K}} \rangle$ *be defined by* $\Omega(\chi)(a^{-1}b) = (\chi(a))^{-1}\chi(b)$*, for all* $a, b \in L$*. (Refer to Lemma 4.5.)*

Definition 4.7. *The functor* $\Omega^{-1} : \mathcal{LG}_{cn} \to \mathcal{ORL}$ *is defined as follows:*

(a) *For every object* $\langle \mathbf{G}, \sigma \rangle$ *of* $\mathcal{LG}_{cn}$*,* $\Omega^{-1}(\langle \mathbf{G}, \sigma \rangle) = \mathbf{G}_\sigma$*. (Recall that* $\mathbf{G}_\sigma$ *denotes the residuated lattice with underlying set the image of* σ*; refer to Lemma 3.1.)*

(b) *For every morphism* $\varphi : \langle \mathbf{G}, \sigma \rangle \to \langle \mathbf{H}, \tau \rangle$ *in the category* $\mathcal{LG}_{cn}$*,* $\Omega^{-1}(\varphi) : \mathbf{G}_\sigma \to \mathbf{H}_\tau$ *is the restriction of* φ *on* $\mathbf{G}_\sigma$*.*

We need an additional auxiliary result.

Lemma 4.8. *For every object* $\langle \mathbf{H}, \tau \rangle$ *in* $\mathcal{LG}_{cn}$*,* $\Omega\Omega^{-1}(\langle \mathbf{H}, \tau \rangle)$ *is isomorphic to* $\langle \mathbf{H}, \tau \rangle$*.*

Proof. Let $\langle \mathbf{H}, \tau \rangle$ be in $\mathcal{LG}_{cn}$ and let $\mathbf{L} = \mathbf{H}_\tau$ (see Lemma 3.1). We need to prove that $\langle \mathbf{H}, \tau \rangle$ is isomorphic to $\langle \mathbf{G}(\mathbf{L}), \sigma_{\mathbf{L}} \rangle$. Now both $\mathbf{H}$ and $\mathbf{G}(\mathbf{L})$ are lattice-ordered groups of quotients of $\mathbf{L}$. Hence, in light of Lemma 4.3, there exists a lattice-ordered group isomorphism $\varphi : \mathbf{H} \longrightarrow \mathbf{G}(\mathbf{L})$ that fixes the elements of $\mathbf{L}$. Hence, it is left to establish that $\varphi\tau = \sigma_{\mathbf{L}}\varphi$. Let $\cdot_1$ and $\cdot_2$ denote the multiplications in $\mathbf{H}$ and $\mathbf{G}(\mathbf{L})$, respectively, and let $^{-1}$ denote inversion in both algebras. Let $a^{-1} \cdot_1 b$ be a representative element of $\mathbf{H}$, with $a, b \in L$. We have,

$\varphi\tau(a^{-1}\cdot_1 b) = \varphi(a\backslash_{\mathbf{L}} b) = a\backslash_{\mathbf{L}} b = \sigma_{\mathbf{L}}(a^{-1}\cdot_2 b) = \sigma_{\mathbf{L}}\varphi(a^{-1}\cdot_1 b)$, where $\backslash_{\mathbf{L}}$ denotes the left division operation in $\mathbf{L}$. Thus, $\varphi\tau = \sigma_{\mathbf{L}}\varphi$, as was to be shown.

□

The proof of the main result is an immediate consequence of the preceding lemmas.

Theorem 4.9. *The pair of functors* $\Omega : \mathcal{ORL} \to \mathcal{LG}_{cn}$ *and* $\Omega^{-1} : \mathcal{LG}_{cn} \to \mathcal{ORL}$ *constitutes an equivalence of the categories* $\mathcal{ORL}$ *and* $\mathcal{LG}_{cn}$.

Proof. Lemma 4.5 ensures that Ω is a functor. By Theorem 1, page 93 of [22], it will suffice to prove the following:

(a) The functor Ω is faithful and full.
(b) For every object $\langle \mathbf{H}, \tau\rangle$ in $\mathcal{LG}_{cn}$, $\Omega\Omega^{-1}(\langle \mathbf{H}, \tau\rangle)$ is isomorphic to $\langle \mathbf{H}, \tau\rangle$.

Recall that Ω is faithful (respectively, full) if for every pair of objects $\mathbf{L}$, $\mathbf{K}$ in $\mathcal{ORL}$, the map $\chi \mapsto \Omega(\chi)$ of $Hom_{\mathcal{ORL}}(\mathbf{L}, \mathbf{K})$ to $Hom_{\mathcal{LG}_{cn}}(\Omega\mathbf{L}, \Omega\mathbf{K})$ is injective (respectively, surjective). Now Condition (b) was proved in Lemma 4.8. With regard to (a), if χ_1 and χ_2 are two distinct morphisms in $Hom_{\mathcal{ORL}}(\mathbf{L}, \mathbf{K})$, then $\Omega(\chi_1)$ and $\Omega(\chi_2)$ are distinct, since they extend χ_1 and χ_2, respectively. This establishes faithfulness. To prove that Ω is also full, let φ be any morphism in $Hom_{\mathcal{LG}_{cn}}(\Omega\mathbf{L}, \Omega\mathbf{K})$. Then its restriction $\Omega^{-1}(\varphi)$ on $\mathbf{L}$ is in $Hom_{\mathcal{ORL}}(\mathbf{L}, \mathbf{K})$, and both φ and $\Omega(\Omega^{-1}(\varphi))$ are morphisms in $Hom_{\mathcal{LG}_{cn}}(\Omega\mathbf{L}, \Omega\mathbf{K})$ that extend $\Omega^{-1}(\varphi)$. Then the uniqueness part of Lemma 4.5 implies that $\varphi = \Omega(\Omega^{-1}(\varphi))$, and hence Ω is surjective.

□

5. Other Categorical Equivalences

Given any subcategory $\mathcal{V}$ of $\mathcal{ORL}$, which is defined by identities relative to $\mathcal{ORL}$, it is easy to specify a subcategory $\mathcal{V}^*$ of $\mathcal{LG}_{cn}$ that is equivalent to $\mathcal{V}$ via the restriction of the functors Ω and Ω^{-1}. Indeed, we can define inductively for every term t in the language of residuated lattices, a term t^* in the language of lattice-ordered groups with an additional unary operator, σ, as follows:

$e^* = e$ and $x^* = \sigma(x)$, for every variable x;
$(r \cdot s)^* = r^* \cdot s^*$;
$(r\backslash s)^* = \sigma(r^{*-1}s^*)$;
$(s/r)^* = \sigma(s^*r^{*-1})$;
$(r \vee s)^* = r^* \vee s^*$; and

$$(r \wedge s)^* = \sigma(r^* \wedge s^*).$$

Then clearly the desired category $\mathcal{V}^*$ is the full subcategory of $\mathcal{LG}_{cn}$ whose objects satisfy all the identities $r^* \approx s^*$ for every identity $r \approx s$ that is valid in $\mathcal{V}$.

In what follows, we will examine this correspondence for a few interesting subclasses of $\mathcal{C}an\mathcal{RL}$.

Let $\mathcal{V}_1$ be the class of all cancellative residuated lattices $\mathbf{L}$ satisfying the condition $Lx = xL$, for all $x \in L$. It is immediate that $\mathcal{V}_1 \subseteq \mathcal{ORL}$ and $\mathcal{V}_1$ is a subvariety of $\mathcal{RL}$. The defining equations for $\mathcal{V}_1$, relative to $\mathcal{RL}$, are $xy/y \approx x \approx y\backslash yx$ and $(xy/x)x \approx xy \approx y(y\backslash xy)$. Thus, in light of Theorem 4.9 and the discussion at the beginning of this section, we have:

Proposition 5.1. *$\mathcal{V}_1$ and $\mathcal{V}_1^*$ are equivalent, with the equivalence being implemented by the restrictions of the functors Ω and Ω^{-1}.* □

Recall that $\mathcal{CC}an\mathcal{RL}$ is the category of commutative, cancellative residuated lattices and residuated lattice homomorphisms, while $\mathcal{CLG}_{cn}$ is the full subcategory of $\mathcal{LG}_{cn}$ consisting of objects, $\langle \mathbf{G}, \sigma\rangle$, whose first components are abelian lattice-ordered groups

Corollary 5.2. *The categories $\mathcal{CC}an\mathcal{RL}$ and $\mathcal{CLG}_{cn}$ are equivalent. The equivalence is implemented by the restrictions of the functors Ω and Ω^{-1}.* □

The proof of the next proposition is more involved.

Proposition 5.3. *Let $\mathcal{V}_2$ be the subcategory of $\mathcal{ORL}$ whose objects satisfy the law*

(5.1) $x(y \wedge z) \approx xy \wedge xz.$

Let $\mathcal{V}_2^*$ be the subcategory of $\mathcal{LG}_{cn}$ whose objects $\langle \mathbf{G}, \sigma\rangle$ satisfy

(5.2) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$*, for all* $x, y \in G$.

Then $\mathcal{V}_2$ and $\mathcal{V}_2^*$ are categorically equivalent. The equivalence is implemented by the restrictions of the functors Ω and Ω^{-1}.

Proof. It will suffice to prove that for all $\langle G, \sigma\rangle \in \mathcal{LG}_{cn}$, $\langle G, \sigma\rangle$ satisfies (5.2) if and only if $\mathbf{G}_\sigma$ satisfies (5.1).

Suppose first that $\langle G, \sigma\rangle \in \mathcal{V}_2^*$ satisfies (5.2). Then the meet of two open elements is open, whence $\mathbf{G}_\sigma$ is a lattice-ordered submonoid of $\mathbf{G}$. But the law (5.1) holds in any lattice-ordered group. It follows that (5.1) holds in $\mathbf{G}_\sigma$ since it holds in $\mathbf{G}$.

Next suppose that $\mathbf{G}_\sigma$ satisfies (5.1). Let $\wedge^{\mathbf{G}}$ and $\wedge$ denote the meet operations in $\mathbf{G}$ and $\mathbf{G}_\sigma$, respectively. To begin with, note that $\wedge$ is the restriction of $\wedge^{\mathbf{G}}$ to $\mathbf{G}_\sigma$. Indeed, let $x, y \in G_\sigma$. It is evident that $x \wedge y$ is a lower bound of x and y in $\mathbf{G}$. Now every element of $\mathbf{G}$ is of the form $a^{-1}b$, for some $a, b \in G_\sigma$. Thus, if such an element is a lower bound of x and y in $\mathbf{G}$, then $b \leq ax$ and $b \leq ay$ in $\mathbf{G}_\sigma$. By (5.1), $b \leq a(x \wedge y)$ in $\mathbf{G}_\sigma$, and so $a^{-1}b \leq x \wedge y$ in $\mathbf{G}$. This shows that $x \wedge y$ is the greatest lower bound of x and y in $\mathbf{G}$.

Hence, if $a^{-1}b, c^{-1}d$ are two representative elements of $\mathbf{G}$, with $a, b, c, d \in G_\sigma$, then the meet of $a^{-1}b$ and $c^{-1}d$ in $\mathbf{G}$ is given by the formula,

(5.3) $(a^{-1}b) \wedge^{\mathbf{G}} (c^{-1}d) = (xa)^{-1}(xb \wedge yd)$,

where x, y are any two elements of G_σ such that $xa = yc$. (Refer to the proof of condition (iii) of Lemma 4.2 and recall that multiplication distributes over meets in any lattice-ordered group.)

Now, in light of Lemma 3.1, the left division operation $\backslash$ in $\mathbf{G}_\sigma$ is given by $a \backslash b = \sigma(a^{-1}b)$, for all $a, b \in G$. Therefore, condition (5.3), together with cancellativity, yields $\sigma(a^{-1}b \wedge^{\mathbf{G}} c^{-1}d) = (xa)\backslash(xb \wedge yd) = (xa\backslash xb) \wedge (xa\backslash yd) = (xa\backslash xb) \wedge (yc\backslash yd) = (a\backslash b) \wedge (c\backslash d) = \sigma(a^{-1}b) \wedge \sigma(c^{-1}d) = \sigma(a^{-1}b) \wedge^{\mathbf{G}} \sigma(c^{-1}d)$. This establishes (5.2) and completes the proof of the proposition.

□

Corollary 5.4. *Any residuated lattice in $\mathcal{ORL}$ that satisfies the law $x(y \wedge z) \approx xy \wedge xz$ can be represented as a residuated lattice of order automorphisms of a chain; multiplication is the usual composition of maps and the lattice operations are defined point-wise. In particular, such a residuated lattice has a distributive lattice reduct.*

Proof. This is a direct consequence of Proposition 5.3 and Holland's representation theorem, [19], which states that every lattice-ordered group can be represented as a lattice-ordered group of ordered automorphisms of a chain, with operations defined as in the statement of the lemma. □

Corollary 5.5. *Let $\mathcal{V}_3$ be the subvariety of $\mathcal{CC}an\mathcal{RL}$ satisfying the law*

(5.1) $x(y \wedge z) \approx xy \wedge xz$.

Let $\mathcal{V}_2^*$ be the subcategory of $\mathcal{CLG}_{cn}$ whose objects $\langle \mathbf{G}, \sigma \rangle$ satisfy

(5.2) $\sigma(x \wedge y) = \sigma(x) \wedge \sigma(y)$, *for all* $x, y \in G$.

Then $\mathcal{V}_3$ and $\mathcal{V}_3^*$ are categorically equivalent. The equivalence is implemented by the restrictions of the functors Ω and Ω^{-1}. □

Proposition 5.6. *Let $\mathcal{V}_4$ be the subcategory of $\mathcal{ORL}$ whose objects satisfy the law*

(5.4) $x\backslash(y \vee z) \approx (x\backslash y) \vee (x\backslash z)$.

Let $\mathcal{V}_4^*$ *be the subcategory of* $\mathcal{LG}_{cn}$ *whose objects* $\langle \mathbf{G}, \sigma\rangle$ *satisfy*

(5.5) $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$, *for all* $x, y \in G$.

Then $\mathcal{V}_4$ *and* $\mathcal{V}_4^*$ *are categorically equivalent. The equivalence is implemented by the restrictions of the functors* Ω *and* Ω^{-1}.

Proof. It will suffice to prove that for all $\langle G, \sigma\rangle \in \mathcal{LG}_{cn}$, $\langle G, \sigma\rangle$ satisfies (5.5) if and only if $\mathbf{G}_\sigma$ satisfies (5.4).

To begin with, recall that, in light of Lemma 3.1, the left division operation $\backslash$ in $\mathbf{G}_\sigma$ is given by $a\backslash b = \sigma(a^{-1}b)$, for all $a, b \in G$. Suppose now that $\langle G, \sigma\rangle$ satisfies (5.5). Then we have, for all elements a, b, c of G_σ, $a\backslash(b\vee c) = \sigma(a^{-1}(b\vee c)) = \sigma((a^{-1}b)\vee(a^{-1}c)) = \sigma(a^{-1}b)\vee\sigma(a^{-1}c) = (a\backslash b) \vee (a\backslash c)$. This establishes (5.4).

Conversely, suppose that $\mathbf{G}_\sigma$ satisfies (5.4), and let $a^{-1}b, c^{-1}d$ be two representative elements of $\mathbf{G}$, with $a, b, c, d \in G_\sigma$. In view of Lemma 4.2, the join of $a^{-1}b$ and $c^{-1}d$ in $\mathbf{G}$ is given by the formula,

$$(a^{-1}b) \vee (c^{-1}d) = (xa)^{-1}(xb \vee yd),$$

where x, y are any two elements of G_σ such that $xa = yc$. Therefore, condition (5.4), together with cancellativity, yields $\sigma(a^{-1}b \vee c^{-1}d) = (xa)\backslash(xb \vee yd) = (xa\backslash xb) \vee (xa\backslash yd) = (xa\backslash xb) \vee (yc\backslash yd) = (a\backslash b) \vee (c\backslash d) = \sigma(a^{-1}b) \vee \sigma(c^{-1}d)$. This establishes (5.5) and completes the proof of the proposition. □

In what follows, we denote by $\mathcal{CCanRepRL}$ the variety of commutative, cancellative representable residuated lattices. This is simply the subvariety of $\mathcal{RL}$ that is generated by all commutative, cancellative totally ordered residuated lattices.

Corollary 5.7. *The variety* $\mathcal{CCanRepRL}$ *is equivalent to the subcategory* $\mathcal{V}_5^*$ *of* $\mathcal{CLG}_{cn}$ *whose objects* $\langle \mathbf{G}, \sigma\rangle$ *satisfy*

(5.5) $\sigma(x \vee y) = \sigma(x) \vee \sigma(y)$, *for all* $x, y \in G$.

The equivalence is implemented by the restrictions of the functors Ω *and* Ω^{-1}.

Proof. This result is an immediate consequence of the preceding proposition and of the fact, established in [2], that a commutative residuated lattice satisfying the identity $x \to x \approx e$ – which clearly holds in any commutative and cancellative residuated lattice – is representable if and only if it satisfies the identity

(5.6) $x \to (y \vee z) \approx (x \to y) \vee (x \to z)$.

□

We note, in connection with Corollary 5.7, that the image of a conucleus σ on an abelian lattice-ordered group $\mathbf{G}$ can be representable without the nucleus being join preserving. Thus, it is essential for the validity of this result that $\mathbf{G}$ be the group of quotients of $\mathbf{G}_\sigma$. The following example illustrates this point. Let $\mathbb{R}$ be the lattice ordered abelian group of reals, and let $\mathbf{G} = \mathbb{R} \times \mathbb{R}$. Let σ be the nucleus on $\mathbf{G}$ defined by $\sigma(x, y) = (x \wedge y, x \wedge y)$, where $x \wedge y = min\{x, y\}$ in $\mathbb{R}$. Then the image $\mathbf{G}_\sigma$ of σ is isomorphic to $\mathbb{R}$, and hence it is representable, but (5.5) does not hold. For instance, $\sigma((0, 1) \vee (1, 0)) = (1, 1)$, but $\sigma(0, 1) \vee \sigma(1, 0) = (0, 0)$.

As was noted in the proof of Corollary 5.7, the law (5.6) implies representability, which clearly implies the law (5.2). Hence, Corollaries 5.5 and 5.7 yield the following result.

Corollary 5.8. *If $\langle \mathbf{G}, \sigma \rangle \in \mathcal{CLG}_{cn}$ satisfies (5.5), then it also satisfies (5.2).* □

Another immediate consequence of Theorem 4.9 and the discussion at the beginning of this section is the following result.

Proposition 5.9. *The subcategory $\mathcal{V}_6$ of $\mathcal{ORL}$ consisting of integral Ore residuated lattices is equivalent to the subcategory $\mathcal{V}_6^*$ of $\mathcal{LG}_{cn}$ whose objects $\langle \mathbf{G}, \sigma \rangle$ satisfy the law $\sigma(x) \leq e$.* □

A more interesting categorical equivalence, refer to Corollary 6.7 of [12], is presented in the next result of this section and concerns the class of cancellative GMV-algebras. An extensive investigation of GMV-algebras has been presented in [12]; refer also to Section 6 below for further discussion regarding their relationship with classical MV-algebras. Proofs of the properties presented below may be found in [2], [5] or [21].

The variety, $\mathcal{GBL}$, of *GBL-algebras* (*generalized BL-algebras*) is the subvariety of $\mathcal{RL}$ defined by the laws

(5.7) $y(y\backslash x \wedge e) \approx x \wedge y \approx (x/y \wedge e)y$.

The variety, $\mathcal{GMV}$, of *GMV-algebras* (*generalized MV-algebras*) is the subvariety of $\mathcal{GBL}$ defined by

(5.8) $x/(y\backslash x \wedge e) = x \vee y = (x/y \wedge e)\backslash x$.

Note that both of these classes include the variety of lattice-ordered groups.

Instead of verifying the identities (5.8), it is often more convenient to verify the equivalent quasi-identities

(5.9) $x \leq y \;\Rightarrow\; y = x/(y\backslash x)$ and $x \leq y \;\Rightarrow\; y = (x/y)\backslash x$.

Likewise, the identities (5.7), are equivalent to the quasi-identities – often referred to as divisibility conditions –

(5.10) $x \leq y \;\Rightarrow\; x = y(y\backslash x)$ and $x \leq y \;\Rightarrow\; x = (x/y)y$.

In light of (5.7) and (5.8), the variety, $\mathcal{IGMV}$, of integral GMV-algebras is defined by the identities

(5.11) $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$,

while the variety, $\mathcal{IGBL}$, of integral GBL-algebras is defined by the identities

(5.12) $y(y\backslash x) \approx x \wedge y \approx (x/y)y$.

Let $\mathbf{L}$ be a residuated lattice. For subalgebras $\mathbf{A}$ and $\mathbf{B}$ of $\mathbf{L}$, the *inner direct product* $\mathbf{A} \otimes \mathbf{B}$ is the lattice join $\mathbf{A} \vee \mathbf{B}$ – taken in the lattice of subalgebras of $\mathbf{L}$ – if the map $(x, y) \mapsto xy$ is an isomorphism from the direct product $\mathbf{A} \times \mathbf{B}$ onto $\mathbf{A} \vee \mathbf{B}$, but is otherwise undefined (see [21]).

A main tool in studying the structure of GBL-algebras and GMV-algebras is the following decomposition result established in [12].

Lemma 5.10. ([12]) *A residuated lattice* $\mathbf{L}$ *is a GMV-algebra (respectively, GBL-algebra) if and only if it has an inner direct product decomposition* $\mathbf{L} = \mathbf{A} \otimes \mathbf{B}$*, where* $\mathbf{A}$ *is an* ℓ*-group and* $\mathbf{B}$ *is an integral GMV-algebra (respectively, integral GBL-algebra).* □

Part (1) of the following lemma was established in [2], while part (2) follows from part (1) and Lemma 5.10.

Lemma 5.11.

(1) *The varieties of cancellative integral GBL-algebras and cancellative integral GMV-algebras coincide, and they are precisely the negative cones of lattice-ordered groups.*
(2) *The varieties of cancellative GBL-algebras and cancellative GMV-algebras coincide. Moreover, a residuated lattice is a cancellative GMV-algebra (equivalently, a cancellative GBL-algebra) if and only if it has an inner direct product decomposition* $\mathbf{L} = \mathbf{A} \otimes \mathbf{B}$*, where* $\mathbf{A}$ *is an* ℓ*-group and* $\mathbf{B}$ *is the negative cone of a lattice-ordered group.* □

Let us denote by $\mathcal{C}an\mathcal{GMV}$ the variety of cancellative GMV-algebras. It is clear that $\mathcal{C}an\mathcal{GMV} \subseteq \mathcal{ORL}$, in fact, $\mathcal{C}an\mathcal{GMV} \subseteq \mathcal{V}_1$.

Proposition 5.12. *The variety* $\mathcal{C}an\mathcal{GMV}$ *is equivalent to the subcategory* $\mathcal{C}an\mathcal{GMV}^*$ *of* $\mathcal{LG}_{cn}$ *whose objects* $\langle \mathbf{G}, \sigma \rangle$ *satisfy*

(5.13) $\sigma(\sigma(x) \wedge y) = \sigma(x) \wedge y$*, for all* $x, y \in G$*.*

The equivalence is implemented by the restrictions of the functors Ω and Ω^{-1}.

Proof. Suppose that $\langle \mathbf{G}, \sigma \rangle$ satisfies (5.13). We claim that $\mathbf{G}_\sigma$ is a GMV-algebra. In view of Lemma 5.11, it will suffice to prove that $\mathbf{G}_\sigma$ satisfies the divisibility conditions (5.10). Note first that the set G_σ of open elements of σ is downward closed, that is, if $x \in G_\sigma$ and $y \leq x$, then $y \in G_\sigma$. It follows that the negative cone G^- of $\mathbf{G}$ is a subset of G_σ, since $e \in G_\sigma$. Next, let $x, y \in G_\sigma$ such that $x \leq y$. Then $y^{-1}x \leq e$ and so $y^{-1}x \in G_\sigma$. It follows that $y\sigma(y^{-1}x) = y(y^{-1}x) = x$. Hence, in particular, $y(y \backslash x) = x$. In a similar fashion, $(x/y)y = x$. Thus, the divisibility conditions (5.10) are satisfied.

Conversely, suppose that $\mathbf{G}_\sigma$ is a GMV-algebra. Then it has an inner direct product decomposition $\mathbf{G}_\sigma = \mathbf{A} \otimes \mathbf{B}^-$, where $\mathbf{A}$ and $\mathbf{B}^-$ are subalgebras of $\mathbf{G}_\sigma$, $\mathbf{A}$ is an ℓ-group and $\mathbf{B}^-$ is the negative cone of a lattice-ordered group. Hence, the lattice ordered group $\mathbf{G}$ is isomorphic to $\mathbf{A} \otimes \mathbf{B}$. Further, the map σ sending an element $ab \in \mathbf{A} \otimes \mathbf{B}$ to $a(b \wedge e) \in \mathbf{A} \otimes \mathbf{B}^- = \mathbf{G}_\sigma$ clearly satisfies (5.13).

□

Corollary 5.13. *The variety, $\mathcal{LG}^-$, of negative cones of lattice-ordered groups is equivalent to the subcategory $(\mathcal{LG}^-)^*$ of $\mathcal{LG}_{cn}$ whose objects $\langle \mathbf{G}, \sigma \rangle$ satisfy*

(5.8) $\sigma(x) = x \wedge e$, *for all* $x \in G$.

The equivalence is implemented by the restrictions of the functors Ω and Ω^{-1}. □

6. Subobjects and Epimorphic Images in $\mathcal{RL}^\times$

In this section, we introduce a categorical framework for placing under a common umbrella results connecting lattice-ordered groups with algebras arising in algebraic logic. More specifically, we show that these algebras are subobjects of lattice-ordered groups or epimorphic images of negative cones of lattice-ordered groups in the category $\mathcal{RL}^\times$. Recall that $\mathcal{RL}^\times$ is the category whose objects are residuated lattices and whose morphisms are monoid homomorphisms that are also residuated maps.

We start with a simple lemma, which is in the folklore of the subject; refer, for example, to Chapter 0, Section 3 of [15].

Lemma 6.1. *Let $\mathbf{P}$ and $\mathbf{Q}$ be partially ordered sets, let $f : \mathbf{P} \to \mathbf{Q}$ be a residuated map and let $f_\star : \mathbf{Q} \to \mathbf{P}$ be the residual of f. We have the following:*

(i) *$f_\star f$ is a closure operator on* $\mathbf{P}$.
(ii) *$ff_\star$ is an interior operator on* $\mathbf{Q}$.
(iii) *$ff_\star f = f$ and $f_\star f f_\star = f_\star$.*
(iv) *f is injective (respectively, surjective) if and only if $f_\star$ is surjective (respectively, injective).*
(v) *Let P_f denote the image of f and let $Q_{f_\star}$ denote the image of $f_\star$. Then the partially ordered sets $\mathbf{P}_f$ and $\mathbf{Q}_{f_\star}$ – with respect to the partial orders of $\mathbf{Q}$ and $\mathbf{P}$, respectively – are isomorphic. More specifically, the restriction of $f_\star$ on P_f is an isomorphism from P_f to $Q_{f_\star}$. Its inverse is the restriction of f on $Q_{f_\star}$.* □

Given a residuated lattice $\mathbf{L}$ – that is, an object in $\mathcal{RL}^\times$ – by a *subobject* of $\mathbf{L}$ we understand a residuated lattice $\mathbf{K}$ such that $K \subseteq L$ and the inclusion map $i : \mathbf{K} \to \mathbf{L}$ is a morphism in $\mathcal{RL}^\times$.

Our first step towards the promised results is Proposition 6.3, which states that the objects of $\mathcal{ORL}$ are subobjects of lattice-ordered groups in the category $\mathcal{RL}^\times$. Restricting our attention to $\mathcal{CC}an\mathcal{RL}$, we obtain the more complete result that the members of $\mathcal{CC}an\mathcal{RL}$ are precisely the subobjects of abelian lattice-ordered groups in the category $\mathcal{RL}^\times$. These results are immediate consequences of Theorem 4.9, Corollary 5.2 and Lemma 6.2 below. The latter shows that the concept of a "subobject" in $\mathcal{RL}^\times$ is equivalent to the concept of interior extraction introduced in Section 3. (Compare with Theorem 3.1.3 in [25].)

Lemma 6.2. *Let $\mathbf{L}$ be a residuated lattice.*

(1) *Let $\mathbf{K}$ be a subobject of $\mathbf{L}$ and let $i_\star$ denote the residual of the inclusion map $i : \mathbf{K} \to \mathbf{L}$. Then the composition $\sigma = ii_\star : \mathbf{L} \to \mathbf{L}$ is a conucleus and $\mathbf{L}_\sigma = \mathbf{K}$ (as algebras).*
(2) *If σ is a conucleus on $\mathbf{L}$, then the inclusion map $i : \mathbf{L}_\sigma \to \mathbf{L}$ is a morphism in $\mathcal{RL}^\times$, that is, $\mathbf{L}_\sigma$ is a subobject of $\mathbf{L}$ in $\mathcal{RL}^\times$.*

Proof. We first establish (1). In light of Lemma 6.1(iv), $i_\star$ is surjective and hence, by Condition (i) of the same lemma, σ is an interior operator on $\mathbf{L}$ with image K. Hence, to prove that σ is a conucleus it will suffice to prove that $\sigma(x)\sigma(y) \le \sigma(xy)$, for all $x, y \in L$. Let $x, y \in L$. We have $\sigma(x)\sigma(y) \le xy$, since σ is an interior operator. By assumption, multiplication in $\mathbf{K}$ coincides with that in $\mathbf{L}$ and hence the relation $\sigma(x)\sigma(y) \le xy$ yields $\sigma(x)\sigma(y) = \sigma(\sigma(x)\sigma(y)) \le \sigma(xy)$. It follows that K is the interior extraction corresponding to the conucleus σ, and hence the structures $\mathbf{K}$ and $\mathbf{L}_\sigma$ are equal in light of Lemma 3.1.

The proof of (2) is immediate, since the inclusion map $i : \mathbf{L}_\sigma \to \mathbf{L}$ is monoid homomorphism and a residuated map with residual the map $\sigma : \mathbf{L} \to \mathbf{L}_\sigma$.

□

Proposition 6.3. *Every Ore residuated lattice is a subobject of a lattice-ordered group in the category $\mathcal{RL}^{\times}$.*

Proof. Theorem 4.9 and Lemma 6.2. □

The following result is an immediate consequence of Corollary 5.2 and Lemma 6.2.

Proposition 6.4. *The variety, $\mathcal{CCanRL}$, of commutative cancellative residuated lattices is the class of all subobjects of abelian lattice-ordered groups in the category $\mathcal{RL}^{\times}$.* □

The framework of the category $\mathcal{RL}^{\times}$ also sheds new light into the main results in [24], [10] and [12], by enabling us to view integral GMV-algebras as the epimorphic images, in $\mathcal{RL}^{\times}$, of negative cones of lattice-ordered groups. MV-algebras and bounded GMV-algebras are special epimorphic images of negative cones of abelian lattice-ordered groups and arbitrary lattice-ordered groups, respectively.

We will need some additional terminology and references to the literature. A *residuated bounded lattice* is an algebraic system $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, 0\rangle$ such that $\langle L, \wedge, \vee, \cdot, \backslash, /, e\rangle$ is a residuated lattice and $\mathbf{L}$ satisfies $x \vee 0 \approx x$. Note that $\top = 0\backslash 0 = 0/0$ is the greatest element of such an algebra.

Commutative, integral residuated bounded lattices have been studied extensively in both algebraic and logical form, and include important classes of algebras, such as the variety of MV-algebras, which provides the algebraic setting for Łukasiewicz's infinite-valued propositional logic. Several term equivalent formulations of MV-algebras have been proposed (see, for example, [6]). Within the context of commutative, residuated bounded lattices, MV-algebras are axiomatized by the identity $(x \to y) \to y \approx x \vee y$, which is a relativized version of the law $\neg\neg x \approx x$ of double negation. The appropriate non-commutative generalization of such an algebra is a residuated bounded lattice that satisfies the identities $x/(y\backslash x) \approx x \vee y \approx (x/y)\backslash x$. These algebras are term equivalent to the algebras considered, among other places, in [10], [13] and [14] under the names GMV-algebras and pseudo-MV-algebras. We use the term *bounded GMV-algebras* for these algebras. The reader will recall that the subvariety of, necessarily integral, residuated lattices that satisfy the preceding law is the variety, $\mathcal{IGMV}$, of integral GMV-algebras.

A fundamental result in the theory of MV-algebras, due to Mundici [24], is the categorical equivalence between the category of MV-algebras and the category of unital abelian lattice-ordered groups, that is, abelian lattice-ordered groups with a designated strong order unit. Dvurečenskij generalized, in [10], the Mundici correspondence to bounded GMV-algebras and arbitrary unital lattice-ordered groups. Dvurečenskij's result is subsumed by the following result in [12].

Lemma 6.5. ([12])

(1) *Let $\mathcal{LG}^{-}_{ncl}$ be the category each object, $\langle \mathbf{B}, \gamma \rangle$, of which consists of the negative cone, $\mathbf{B}$, of a lattice-ordered group augmented with a nucleus γ on it whose image generates $\mathbf{B}$ as a monoid. Let the morphisms of these categories be algebra homomorphisms. Then the categories $\mathcal{IGMV}$ and $\mathcal{LG}^{-}_{ncl}$ are equivalent.*
(2) *If $\mathbf{L}$ is an integral GMV-algebra and γ is a nucleus on $\mathbf{L}$, then $\mathbf{L}_\gamma$ is an integral GMV-algebra.* □

The connection of this result with surjective morphisms in $\mathcal{RL}^\times$ is provided by the following result, which shows that all closure retracts of a residuated lattice $\mathbf{L}$ are of the form $\mathbf{L}_\gamma$ for some nucleus γ on $\mathbf{L}$, where $\mathbf{L}_\gamma$ is the residuated lattice defined in Lemma 3.2. (Compare with Theorem 3.1.1 of [25].)

Lemma 6.6. *Let $f : \mathbf{L} \to \mathbf{K}$ be a surjective morphism in $\mathcal{RL}^\times$. Then there exists a nucleus γ on $\mathbf{L}$ such that $\mathbf{K} \cong \mathbf{L}_\gamma$.*

Proof. Let $f_\star$ be the residual of f and let $\gamma = f_\star f$ be the associated closure operator on $\mathbf{L}$ (Lemma 6.1). To prove that γ is a nucleus, we need to show that $\gamma(a)\gamma(b) \leq \gamma(ab)$, that is, $(f_\star f(a))(f_\star f(b)) \leq (f_\star f)(ab)$, for all $a, b \in L$. Let $a, b \in L$. Since f preserves multiplication and $f = f f_\star f$, by Lemma 6.1, we have the following equivalences.

$$
\begin{aligned}
(f_\star f(a))(f_\star f(b)) \leq f_\star f(ab) &\iff f((f_\star f(a))(f_\star f(b))) \leq f(ab) \\
&\iff (f f_\star f(a))(f f_\star f(b)) \leq f(ab) \\
&\iff f(a)f(b) \leq f(ab)
\end{aligned}
$$

Therefore, γ is a nucleus.

Now, since f is surjective, by Lemma 6.1(v), $f_\star$ is an isomorphism of the partially ordered sets $\mathbf{K}$ and $\mathbf{L}_\gamma$. Thus, to prove that $\mathbf{K} \cong \mathbf{L}_\gamma$, it will suffice to show that $f_\star : \mathbf{K} \to \mathbf{L}_\gamma$ is a monoid homomorphism. Note first that $f_\star$ preserves the multiplicative identities. Further, we have for any $a, b \in L$,

$$\begin{aligned} f_\star(f(a)f(b)) &= f_\star(f(ab)) \\ &= (f_\star f)(ab) \\ &= f_\star f(a) \circ_\gamma f_\star f(b) \\ &= \gamma(a) \circ_\gamma \gamma(b) \end{aligned}$$

Therefore, $K \cong L_\gamma$, as was to be shown. □

Combining the last two results we get:

Proposition 6.7. *A residuated lattice is an integral GMV-algebra if and only if it is the epimorphic image, in $\mathcal{RL}^\times$, of the negative cone of a lattice-ordered group.* □

We note that bounded GMV-algebras, and in particular MV-algebras, are images of special nuclei. More specifically, they are of the form $\mathbf{B}_{\gamma_a}$, where $\mathbf{B}$ is the negative cone of a lattice-ordered group, a is a fixed element of B and γ_a is the nucleus on $\mathbf{B}$ defined by $\gamma_a(x) = a \vee x$, for all $x \in L$ (see [12] for details).

7. Residuated Lattices as Subobjects of Involutive Residuated Lattices

This section of the paper is concerned with the question of whether the results of the previous sections can be extended to residuated lattices that are not cancellative or weakly cancellative. (Refer to the last section for a stronger result involving weakly cancellative residuated lattices.) In this setting, an appropriate substitute for the concept of a lattice-ordered group is that of an involutive residuated lattice. By employing an embedding result in [26] (see also [25] and [3]), we show that every residuated lattice with top element is a subobject, in $\mathcal{RL}^\times$, of an involutive residuated lattice. It's an open question at this time as to whether this correspondence extends to a categorical equivalence.

An *involutive residuated lattice* is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \ ', e\rangle$ such that

(i) $\langle L, \wedge, \vee\rangle$ is a lattice;
(ii) $\langle L, \cdot, e\rangle$ is a monoid;
(iii) the unary operation $'$ is an involution of the lattice $\langle L, \wedge, \vee\rangle$, that is, a dual automorphism such that $x'' = x$, for all $x \in L$; and
(iv) $xy \le z \iff y \le (z'x)' \iff x \le (yz')'$, for all $x, y, z \in L$.

The term "involutive residuated lattice" is suggestive of the fact that multiplication is residuated in such an algebra. Indeed, it is immediate, from condition (iv) above, that for all elements $x, y \in L, x\backslash y = (y'x)'$ and $y/x = (xy')'$.

It is routine to verify that the class, $\mathcal{I}n\mathcal{RL}$, of involutive residuated lattices is a finitely based variety. Involutive residuated lattices have received considerable attention both from the logic and algebra communities. From a logical perspective, they are the algebraic counterparts of the propositional non-commutative linear logic without exponentials. From an algebraic perspective, they include a number of important classes of algebras, such as Boolean algebras, MV-algebras and lattice-ordered groups.

It is often convenient to use a term-equivalent description of involutive residuated lattices. Namely, think of them as algebras $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ such that:

(i) $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /\rangle$ is a residuated lattice; and
(ii) d is an *involutive* element. The second condition means that, for all $x \in L$, $d/x = x\backslash d$ (d is *cyclic*) and $d/(x\backslash d) = (d/x)\backslash d = x$ (d is *weakly involutive*).

Note that if $\mathbf{L}_d = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ is an algebra as defined above and we define $x' = d/x$, for all $x \in L$, then $\mathbf{L}' = \langle L, \wedge, \vee, \cdot, {}', e\rangle$ becomes an involutive residuated lattice. On the other hand, if $\mathbf{L}' = \langle L, \wedge, \vee, \cdot, {}', e\rangle$ is an involutive residuated lattice, then the algebra $\mathbf{L}_d = \langle L, \wedge, \vee, \cdot, \backslash, /, e, d\rangle$ – defined by (a) $d = e'$; and (b) $x\backslash z = (z'x)'$, $z/x = (xz')'$, for all $x, z \in L$ – satisfies conditions (i) and (ii) above.

Lemma 7.1. *Let* $\mathbf{L}$ *be a residuated lattice with greatest element* $\top$*.*

(i) $\tilde{\mathbf{L}} = \langle L \times L, \wedge, \vee, \cdot, \backslash, /, E, D\rangle$ *is – with the operations defined below – a residuated lattice with an involutive element* D*:*

$$
\begin{aligned}
(a, x) \wedge (b, y) &= (a \wedge b, x \vee y) \\
(a, x) \vee (b, y) &= (a \vee b, x \wedge y) \\
(a, x)(b, y) &= (ab, y/a \wedge b\backslash x) \\
(a, x)\backslash(b, y) &= (a\backslash b \wedge x/y, ya) \\
(a, x)/(b, y) &= (a/b \wedge x\backslash y, bx) \\
E &= (e, \top) \\
D &= (\top, e)
\end{aligned}
$$

(ii) *That* $D = (\top, e)$ *is involutive follows from the equality* $(a,x)\backslash(\top,e) = (x,a) = (\top,e)/(a,x)$, *for all* $a, x \in L$.

(iii) *Let* $\mathbf{L}^\star = \langle L^\star, \vee, \wedge, \cdot, \backslash^\star, /^\star, E\rangle$, *where*

$$\begin{aligned} L^\star &= L \times \{\top\} \\ B/^\star A &= B/A \wedge (\top, \top), \\ A\backslash^\star B &= A\backslash B \wedge (\top, \top). \end{aligned}$$

Then the map $\varepsilon : \mathbf{L} \to \mathbf{L}^\star$, *defined by* $\varepsilon(a) = (a, \top)$ *for all* $a \in L$, *is a residuated lattice isomorphism.* □

The operations of $\tilde{\mathbf{L}}$ are admittedly confusing at first sight. However, they are quite intuitive if they are viewed in the context of actions of residuated lattices on partially ordered sets. The reader is referred to [26] for details. What is important to keep in mind here is that $\mathbf{L}$ can be identified with $\mathbf{L}^\star$ within $\tilde{\mathbf{L}}$, which is a dualizing residuated lattice and hence an involutive residuated lattice.

Proposition 7.2. *Every residuated lattice with a top element is a subobject in* $\mathcal{RL}^\times$ *of an involutive residuated lattice.*

Proof. Let $\mathbf{L}$ be a residuated lattice with a top element $\top$ and let $\mathbf{L}^\star$ and $\tilde{\mathbf{L}}$ be defined as in Lemma 7.1. In light of Condition (iii) of the same lemma, it will suffice to verify that $\mathbf{L}^\star$ is a subobject of $\tilde{\mathbf{L}}$, which means that the inclusion map $i : \mathbf{L}^\star \to \tilde{\mathbf{L}}$ is a morphism in $\mathcal{RL}^\times$. Thus we have to verify that:

(a) $\mathbf{L}^\star$ is a submonoid of $\tilde{\mathbf{L}}$; and
(b) i is residuated.

The proof of (a) is immediate, since, for all $a, b \in L, (a,\top)(b,\top) = (ab, \top/a \wedge b\backslash\top) = (ab, \top)$. The last equality follows from the fact that $\top/c = c\backslash\top = \top$ in $\mathbf{L}$, for all $c \in L$. To verify (b), consider the map $\sigma : \tilde{\mathbf{L}} \to \mathbf{L}^\star$ – defined by $\sigma(a,x) = (a,\top)$, for all $a, x \in L$. We claim that σ is the residual of i and, hence, the conucleus associated with $\mathbf{L}^\star$. This is again straightforward. Note first that for all $(a,x), (b,y) \in \tilde{L}$, $(a,x) \le (b,y)$ in $\tilde{\mathbf{L}}$ if and only if $a \le b$ and $x \ge y$ in $\mathbf{L}$. Thus, for elements $(a,\top) \in L^\star$ and $(b,y) \in \tilde{L}$,

$$\begin{aligned} i(a,\top) \le (b,y) \text{ in } \tilde{\mathbf{L}} &\iff (a,\top) \le (b,y) \text{ in } \tilde{\mathbf{L}} \\ &\iff a \le b \text{ in } \mathbf{L} \\ &\iff (a,\top) \le (b,\top) \text{ in } \mathbf{L}^\star \\ &\iff (a,\top) \le \sigma(b,y) \text{ in } \mathbf{L}^\star \end{aligned}$$

This completes the proof of (b) and of the proposition. □

It should be noted that the subalgebra of $\tilde{\mathbf{L}}$ generated by $\mathbf{L}^\star$ may be properly contained in $\tilde{\mathbf{L}}$. Thus, verifying that this subalgebra is uniquely determined by $\mathbf{L}^\star$ would be an important first step in producing a categorical equivalence similar to the ones described in earlier sections.

8. Applications to Many-Valued Logic

Throughout this section, we will depart from our standard convention and denote the multiplicative identity of a residuated lattice by 1.

A *ΠMTL-algebra* is a residuated bounded lattice (see section 6) $\mathbf{L} = \langle L, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ that is commutative, integral, representable and satisfies the equation

$$(x \to 0) \vee ((x \to xy) \to y) = 1. \tag{7.1}$$

A *product algebra* is a divisible ΠMTL-algebra.

Product algebras and ΠMTL-algebras have been investigated in the context of many-valued logic; refer, for example, to [16], [17], [20] and [23]. It has been shown in [17] that the variety, $\mathcal{PA}$, of product algebras is generated by the standard product algebra $\mathbf{L} = \langle [0,1], \wedge, \vee, \cdot, \to, 1, 0\rangle$, where multiplication is the usual multiplication of reals and the division operation (residual) is given by

$$a \to b = \begin{cases} \frac{b}{a} & \text{if } \ b < a, \\ 1 & \text{if } \ a \leq b. \end{cases}$$

The variety, $\Pi\mathcal{MTL}$, of ΠMTL-algebras is generated by the class of all *semicancellative* left-continuous t-norms, that is, those t-norms that satisfy the cancellation law for non-zero elements.

Let $\mathbf{L}$ be a subdirectly irreducible ΠMTL-algebra and let K denote the set of non-zero elements of L: $K = L - \{0\}$. Since $\mathbf{L}$ is totally ordered, (7.1) easily implies that K is closed under all the operations of $\mathbf{L}$ – other than 0, of course – and the resulting residuated lattice $\mathbf{K}$ is cancellative. Hence, if $\mathbf{L}$ is a subdirectly irreducible product algebra, then, in light of Lemma 5.11, $\mathbf{K}$ is then negative cone of an lattice-ordered abelian group.

The aforementioned relationship between ΠMTL-algebras and integral members of $\mathcal{CCanRepRL}$, as well as the relationship between product algebras and lattice ordered abelian groups suggests the possibility of establishing a categorial equivalence between ΠMTL-algebras and product algebras with a conucleus. The main result of this section, Theorem 8.11, demonstrates that this is indeed the case.

Given a ΠMTL-algebra $\mathbf{A}$, we can construct a product algebra $\mathbf{A}^*$ in the following manner. First, we represent $\mathbf{A}$ as a subdirect product of subdirectly irreducible (hence totally ordered) ΠMTL-algebras $(\mathbf{A}_i : i \in I)$. Then, for each $i \in I$, the set, C_i, of non-zero elements of $\mathbf{A}_i$ is the subuniverse of an integral member, $\mathbf{C}_i$, of $\mathcal{CC}an\mathcal{R}ep\mathcal{RL}$. It follows that each $\mathbf{C}_i$ can be associated with the totally ordered abelian group, $\mathbf{G}_i$, of its (left) quotients. Now each negative cone $\mathbf{G}_i^-$ of $\mathbf{G}_i$, augmented with a zero element 0_i, gives rise to a product algebra $\mathbf{A}_i^*$, by letting $0_i x = x 0_i = 0_i$, $0_i \to_i x = 1_i$, and $x \to_i 0_i = 0_i$ for $x \neq 0_i$. Let $\mathbf{D}$ be the product of all the algebras $\mathbf{A}_i^*$. Evidently, $\mathbf{D}$ is a product algebra, with implication $\to^*$ defined, for all $x, y \in \mathbf{A}^*$, by

$$(x \to^* y)_i = \begin{cases} ({x_i}^{-1} y_i) \wedge 1_i & \text{if } \ x_i, y_i > 0_i; \\ 1_i & \text{if } \ x_i = 0_i; \text{ and} \\ 0_i & \text{if } \ x_i \neq 0_i \text{ and } y_i = 0_i. \end{cases}$$

Note that $x \to^* y = x \to^* (x \wedge y)$; therefore we will always assume that $y \leq x$ whenever we write $(x \to^* y)$.

With reference to the preceding construction, we will denote by $\mathbf{A}^*$ the subalgebra of $\mathbf{D}$ generated by $\mathbf{A}$.

The following result is immediate.

Lemma 8.1.

(a) $\mathbf{A}^*$ *is a product algebra,* $A \subseteq A^*$*, and* A *is closed with respect to the lattice and monoid operations of* $\mathbf{A}^*$*.*
(b) $\mathbf{A}^*$ *is generated by* A *as a product algebra.*

□

If $\mathbf{A}$ is a ΠMTL-algebra and $\mathbf{A}^*$ is a product algebra satisfying conditions (a) and (b) of Lemma 8.1, then we will say that $\mathbf{A}^*$ is a *product algebra generated by* $\mathbf{A}$.

We will prove below that any such algebra is isomorphic to the concrete algebra $\mathbf{A}^*$ constructed above. In the sequel, whenever $\mathbf{A}$ is a ΠMTL-algebra and $\mathbf{A}^*$ is the product algebra generated by $\mathbf{A}$, the operation symbols without superscript will refer to $\mathbf{A}$ while those with the superscript * will refer to $\mathbf{A}^*$.

Let $\mathbf{A}^*$ be a product algebra generated by a ΠMTL-algebra $\mathbf{A}$, and let us represent $\mathbf{A}^*$ as a subdirect product of a family of totally ordered product algebras $(\mathbf{A}_i^* : i \in I)$. Then for $i \in I$, $\mathbf{A}_i^* = \mathbf{A}^*/P_i^*$ for some prime filter P_i^* of $\mathbf{A}^*$. Let $P_i = P_i^* \cap A$.

Lemma 8.2. *Maintaining the notation established in the preceding paragraph, we have the following for all $i \in I$.*

(i) *P_i is a prime filter of $\mathbf{A}$.*
(ii) *$\mathbf{A}_i = \mathbf{A}/P_i$ is a totally ordered ΠMTL-algebra.*
(iii) *$\mathbf{A}$ is a subdirect product of the family $(\mathbf{A}_i : i \in I)$.*
(iv) *The lattice ordered monoid reduct of $\mathbf{A}_i$ is (isomorphic to) a subreduct of $\mathbf{A}_i^*$.*
(v) *$\mathbf{A}_i^*$ is generated by (the isomorphic image of) $\mathbf{A}_i$ as a product algebra.*

Proof. The proofs of (i), (ii) and (iii) are immediate. With regard to (iv), note that the map $a/P_i \to a/P_i^*$ is a lattice ordered monoid embedding of $\mathbf{A}_i$ into $\mathbf{A}_i^*$. Finally, (v) follows from (iv) and from the fact that $\mathbf{A}^*$ is generated by $\mathbf{A}$ as a product algebra. □

For the remainder of this section, we will use the notation $\neg x$ for $x \to 0$.

Lemma 8.3. *Let $\mathbf{A}$ be a ΠMTL-algebra and $\mathbf{A}^*$ be a product algebra generated by $\mathbf{A}$. Then:*

(a) *The domain of $\mathbf{A}^*$ is the set of all elements of the form $a \to^* b$ with $a, b \in \mathbf{A}$ and $b \leq a$.*
(b) *Consider the term $t(x, y, z, u) =$*
$(\neg x \wedge (z \to u)) \vee (\neg z \wedge (x \to y)) \vee (\neg\neg x \wedge \neg\neg z \wedge (xu \leftrightarrow yz))$.
Then for all $a, b, c, d \in \mathbf{A}$ with $b \leq a$ and $d \leq c$,
$a \to^* b = c \to^* d$ iff $t(a, b, c, d) = 1$ in $\mathbf{A}$.

Proof. Throughout the proof we will fix a subdirect decomposition of $\mathbf{A}^*$ in terms of a family $(\mathbf{A}_i^* : i \in I)$ of totally ordered product algebras. In light of Lemma 8.2, this induces a subdirect representation of $\mathbf{A}$ by means of a family $(\mathbf{A}_i : i \in I)$ of totally ordered ΠMTL-algebras such that for every $i \in I$, $\mathbf{A}_i^*$ is a product algebra generated by $\mathbf{A}_i$.

We first establish (a). Let $B = \{a \to^* b : a, b \in \mathbf{A}, b \leq a\}$. We need to prove that $B = A^*$.

Claim 1
For all $a \to^* b, c \to^* d \in B$, $(a \to^* b) \vee^* (c \to^* d) = (ac \to^* (cb \vee ad))$. Thus B is closed under $\vee^*$.

Proof of Claim 1. Let $i \in I$.

If $a_i = 0_i$ or $c_i = 0_i$, we have $((a \to^* b) \vee^* (c \to^* d))_i = (ac \to^* (cb \vee ad))_i = 1_i$.

If $a_i, c_i \neq 0_i$ and $b_i = 0_i$, then $((a \to^* b) \vee^* (c \to^* d))_i = (c \to^* d)_i$ and $(ac \to^* (cb \vee ad))_i = (ac \to^* ad)_i = (c \to^* d)_i$. Similarly, if $a_i, c_i \neq 0_i$

and $d_i = 0_i$, then $((a \to^* b) \vee^* (c \to^* d))_i = (ac \to^* (cb \vee ad))_i = (a \to^* b)_i$.

Finally, if $a_i, c_i, b_i, d_i \neq 0_i$, then recalling that $\mathbf{A}_i^* \backslash \{0_i\}$ is the negative cone $\mathbf{G}_i^-$ of a totally ordered abelian group $\mathbf{G}_i$, we obtain successively $((a \to^* b) \vee^* (c \to^* d))_i = (a_i c_i)^{-1}(c_i b_i \vee a_i d_i) = (ac \to^* (cb \vee ad))_i$.

This concludes the proof of Claim 1.

Claim 2
B is closed under products and meets in $\mathbf{A}^*$.

Proof of Claim 2. Define, for all $a, b, c, d \in \mathbf{A}$ with $b \leq a$ and $d \leq c$:
$t_{1,1}(a,c) = \neg a \vee \neg c \vee ac, \quad t_{1,2}(a,b,c,d) = \neg\neg a \wedge \neg\neg c \wedge bd,$
$t_1(a,b,c,d) = t_{1,1}(a,c) \to^* t_{1,2}(a,b,c,d).$
$t_{2,1}(a,c) = \neg a \vee \neg c \vee ac, \quad t_{2,2}(a,b,c,d) = \neg\neg a \wedge \neg\neg c \wedge cb \wedge ad,$
$t_2(a,b,c,d) = t_{2,1}(a,c) \to^* t_{2,2}(a,b,c,d),$
$t_{3,1}(a,c) = \neg\neg a \vee c, \quad t_{3,2}(a,d) = \neg a \wedge d,$
$t_3(a,c,d) = t_{3,1}(a,c) \to^* t_{3,2}(a,d),$
$t_{4,1}(a,c) = \neg\neg c \vee a, \quad t_{4,2}(b,c) = \neg c \wedge b,$
$t_4(a,b,c) = t_{4,1}(a,c) \to^* t_{4,2}(b,c).$

One can check by a straightforward computation that, for all $i \in I$,

If $a_i \neq 0_i$ and $c_i \neq 0_i$, then
$t_1(a,b,c,d)_i = (ac \to^* bd)_i = ((a \to^* b)^* \cdot^* (c \to^* d))_i$,
$t_2(a,b,c,d)_i = (ac \to^* (cb \wedge ad))_i = ((a \to^* b)^* \wedge^* (c \to^* d))_i$, and $t_3(a,c,d)_i = t_4(a,b,c)_i = 0_i$;

if $a_i = 0_i$ and $c_i \neq 0_i$, then $t_1(a,b,c,d)_i = t_2(a,b,c,d)_i = t_4(a,b,c)_i = 0_i$, and $t_3(a,c,d)_i = (c \to^* d)_i$;

if $a_i \neq 0_i$ and $c_i = 0_i$, then $t_1(a,b,c,d)_i = t_2(a,b,c,d)_i = t_3(a,c,d)_i = 0_i$, and $t_4(a,b,c)_i = (a \to^* b)_i$; and

if $a_i = c_i = 0$, then $t_3(a,c,d)_i = t_4(a,b,c)_i = 1_i$, and
$((a \to^* b)^* \cdot^* (c \to^* d))_i = ((a \to^* b)^* \wedge^* (c \to^* d))_i = 1_i$.

By Claim 1, B is closed under $\vee^*$. Thus the formulas below establish closure with respect to $\cdot^*$ and $\wedge^*$.

$$(a \to^* b)^* \cdot^* (c \to^* d) = t_1(a,b,c,d) \vee^* t_3(a,c,d) \vee^* t_4(a,b,c)$$
$$(a \to^* b)^* \wedge^* (c \to^* d) = t_2(a,b,c,d) \vee^* t_3(a,c,d) \vee^* t_4(a,b,c)$$

Claim 3
B is closed under $\to^*$.

Proof of Claim 3. Let $(a \to^* b), (c \to^* d) \in A^*$ and let $i \in I$.

We first check that if $a_i, b_i \neq 0_i$, then

$$((a \to^* b)^* \to^* (c \to^* d))_i = (bc \to^* ad)_i.$$

If $c_i, d_i \neq 0_i$, then $((a \to^* b)^* \to^* (c \to^* d))_i = (a_i^{-1}b_i)^{-1}c_i^{-1}d_i = (b_ic_i)^{-1}a_id_i = (bc \to^* ad)_i$. If on the other hand $c_i = 0_i$, then $(c \to^* d)_i = ((a \to^* b)^* \to^* (c \to^* d))_i = (bc \to^* ad)_i = 1_i$. If $c_i \neq 0_i$ and $d_i = 0_i$, then $((a \to^* b)^* \to^* (c \to^* d))_i = (bc \to^* ad)_i = 0_i$.

Now note that if $a_i = 0_i$, then $((a \to^* b)^* \to^* (c \to^* d))_i = (c \to^* d)_i$, and that if $a_i \neq 0_i$ and $b_i = 0_i$, then $((a \to^* b)^* \to^* (c \to^* d))_i = 1_i$.

Next we define, for $a, b, c, d \in \mathbf{A}$ with $b \leq a$ and $d \leq c$:

$s_{1,1}(a,b,c) = \neg a \vee \neg b \vee bc, \quad s_{1,2}(a,b,d) = \neg\neg a \wedge \neg\neg b \wedge ad,$
$s_1(a,b,c,d) = s_{1,1}(a,b,c) \to^* s_{1,2}(a,b,d),$
$s_{2,1}(a,c) = \neg\neg a \vee c, \quad s_{2,2}(a,d) = \neg a \wedge d,$
$s_2(a,c,d) = s_{2,1}(a,c) \to^* s_{2,2}(a,d),$
$s_3(a,b) = \neg b \wedge \neg\neg a.$

Note that for all $i \in I$:

If $a_i, b_i \neq 0_i$, $s_1(a,b,c,d)_i = (bc \to^* ad)_i = ((a \to^* b)^* \to^* (c \to^* d))_i$, and $s_2(a,c,d)_i = s_3(a,b)_i = 0_i$.

If $a_i = 0$, then $s_2(a,c,d)_i = ((a \to^* b)^* \to^* (c \to^* d))_i = (c \to^* d)_i$, and $s_1(a,b,c,d)_i = s_3(a,b)_i = 0_i$.

If $a_i \neq 0_i$ and $b_i = 0_i$, then $s_3(a,b)_i = ((a \to^* b)^* \to^* (c \to^* d))_i = 1_i$.

It follows that

$$(a \to^* b)^* \to^* (c \to^* d) = s_1(a,b,c,d) \vee^* s_2(a,c,d) \vee^* s_3(a,b).$$

Since B is closed under $\vee^*$, $(a \to^* b)^* \to^* (c \to^* d) \in B$, completing the proof of Claim 3 and Case (a) in the statement of the lemma.

It remains to prove (b). Let $i \in I$. Suppose first that either $a_i = 0_i$ or $c_i = 0_i$. Then $(a \to^* b)_i = (c \to^* d)_i$ iff they are both equal to 1_i, that is, iff $a_i = b_i$ and $c_i = d_i$. Moreover since $(\neg\neg a \wedge \neg\neg c \wedge (ad \leftrightarrow bc))_i = 0_i$, we have that $t(a,b,c,d)_i = 1_i$ iff either $a_i = 0_i$ (hence $b_i = 0_i$) and $c_i = d_i$, or $c_i = 0_i$ (hence $d_i = 0_i$) and $a_i = b_i$. Thus $t(a,b,c,d)_i = 1_i$ iff $a_i = b_i$ and $c_i = d_i$ iff $(a \to^* b)_i = (c \to^* d)_i$.

If $a_i, c_i \neq 0_i$, then distinguish the following cases.

If $b_i = d_i = 0_i$, then $t(a,b,c,d)_i = 1_i$ and $(a \to^* b)_i = (c \to^* d)_i = 0_i$.

If $b_i = 0_i$ and $d_i \neq 0_i$, then $t(a,b,c,d)_i = 0_i \neq 1_i$, $(a \to^* b)_i = 0_i$ and $(c \to^* d)_i \neq 0_i$, hence $(a \to^* b)_i \neq (c \to^* d)_i$.

For $d_i = 0_i$ and $b_i \neq 0_i$, the argument is similar.

If $b_i \neq 0_i$ and $d_i \neq 0_i$, then $(a \to^* b)_i = (c \to^* d)_i$ iff $a_i d_i = b_i c_i$. On the other hand, $(((\neg a \wedge (c \to d)) \vee (\neg c \wedge (a \to b)))_i = 0_i$ and $(\neg\neg a \wedge \neg\neg c)_i = 1_i$, therefore $t(a,b,c,d)_i = 1_i$ iff $a_i d_i = b_i c_i$.

The proof of Lemma 8.3 is now complete. □

Lemma 8.4. *Let* $\mathbf{A}$ *be a* ΠMTL*-algebra. If both* $\mathbf{A}^*$ *and* $\mathbf{B}^*$ *are product algebras generated by* $\mathbf{A}$*, then they are isomorphic. Hence they are both isomorphic to the concrete product algebra constructed at the beginning of the section.*

Proof. We will use the superscripts ${}^*_A$ and ${}^*_B$ for the operations of $\mathbf{A}^*$ and $\mathbf{B}^*$, respectively. Every element of $\mathbf{A}^*$ can be written as $a \to^*_A b$, for some $a, b \in \mathbf{A}$, and, likewise, every element of $\mathbf{B}^*$ can be written as $c \to^*_B d$ for some $c, d \in \mathbf{A}$. Set $\Phi(a \to^*_A b) = a \to^*_B b$. We claim that Φ is well defined and an isomorphism from $\mathbf{A}^*$ to $\mathbf{B}^*$. First of all, if $a \to^*_A b = c \to^*_A d$, then by Lemma 8.3 (b), $t(a,b,c,d) = 1$ holds in $\mathbf{A}$, therefore by Lemma 8.3 (b) again, $\Phi(a \to^*_A b) = a \to^*_B b = c \to^*_B d = \Phi(c \to^*_A d)$. Thus, Φ is well-defined. A similar argument shows that Φ is one-one. That Φ is onto is clear. Now we prove that Φ preserves the operations. We start by noting that Φ preserves joins. Indeed, $\Phi((a \to^*_A b) \vee^*_A (c \to^*_A d)) = \Phi(ac \to^*_A (bc \vee ad)) = ac \to^*_B (bc \vee ad) = (a \to^*_B b) \vee^*_B (c \to^*_B d) = \Phi(a \to^*_A b) \vee^*_B \Phi(c \to^*_A d)$.

Moreover, with reference to the notation of the proof of Lemma 8.3, $(a \to^*_A b) \cdot^*_A (c \to^*_A d)$ is the join (in $\mathbf{A}^*$) of $t_{1,1}(a,c) \to^*_A t_{1,2}(a,b,c,d)$, $t_{3,1}(a,c) \to^*_A t_{3,2}(a,d)$ and $t_{4,1}(a,c) \to^*_A t_{4,2}(b,c)$. Since Φ is join-preserving, $\Phi((a \to^*_A b) \cdot^*_A (c \to^*_A d))$ is the join in $\mathbf{B}^*$ of the following elements: $t_{1,1}(a,c) \to^*_B t_{1,2}(a,b,c,d)$, $t_{3,1}(a,c) \to^*_A t_{3,2}(a,d)$ and $t_{4,1}(a,c) \to^*_B t_{4,2}(b,c)$. Again by the proof of Lemma 8.3, this join is $(\Phi(a) \to^*_B \Phi(b)) \cdot^*_B (\Phi(c) \to^*_B \Phi(d))$. We have shown that Φ preserves multiplication. The proof that Φ preserves meet and the residual is quite similar. □

If $\mathbf{A}$ is a ΠMTL-algebra and $\mathbf{A}^*$ is the product algebra generated by $\mathbf{A}$, we define the assignment $\sigma_{\mathbf{A}} : \mathbf{A}^* \to \mathbf{A}^*$ by $\sigma_{\mathbf{A}}(x \to^* y) = x \to y$, for all $x \to^* y \in \mathbf{A}^*$. We reiterate that, following the convention adopted earlier, $\to^*$ denotes the residual in $\mathbf{A}^*$ and $\to$ denotes the residual in $\mathbf{A}$.

Lemma 8.5. *Maintaining the notation of the preceding paragraph, we have the following:*

(i) $\sigma_{\mathbf{A}}$ *is a well-defined map with image* A*.*
(ii) $\sigma_{\mathbf{A}}$ *is a conucleus on* $\mathbf{A}^*$*.*
(iii) $\sigma_{\mathbf{A}}$ *is a lattice endomorphism of* $\mathbf{A}^*$*.*

Proof. (i). We will work with a subdirect decomposition of $\mathbf{A}^*$ in terms of totally ordered product algebras $(A_i^* : i \in I)$. In order to show that $\sigma_{\mathbf{A}}$ is well defined, it suffices to prove that, for all $i \in I$, if $(a \to^* b)_i = (c \to^* d)_i$, then $(a \to b)_i = (c \to d)_i$. Fix an i. If $a_i, b_i, c_i, d_i \neq 0_i$, then $(a \to^* b)_i = a_i^{-1} b_i$ and $(c \to^* d)_i = c_i^{-1} d_i$. Hence, $(a \to^* b)_i = (c \to^* d)_i$ implies $b_i c_i = a_i d_i$, and also $(a \to b)_i = (ac \to bc)_i = (ac \to ad)_i = (c \to d)_i$. If some of a_i, b_i, c_i, d_i is equal to 0_i, then either $(a \to^* b)_i = (c \to^* d)_i = 0_i$ or $(a \to^* b)_i = (c \to^* d)_i = 1_i$. Then the claim follows from the fact that for all $x, y \in \mathbf{A}$ one has: $(x \to^* y)_i = 1_i$ iff $(x \to y)_i = 1_i$ iff $x_i \leq y_i$, and $(x \to^* y)_i = 0_i$ iff $(x \to y)_i = 0_i$ iff $x_i \neq 0_i$ and $y_i = 0_i$. That the image of $\sigma_{\mathbf{A}}$ is A is clear.

(ii) The definition of $\sigma_{\mathbf{A}}$ implies that for $x = a \to^* b \in \mathbf{A}^*$, $\sigma_{\mathbf{A}}(x)$ is the greatest element of $\mathbf{A}$ which is less than or equal to x, therefore $\sigma_{\mathbf{A}}$ is an interior operator. Moreover, the interior extraction corresponding to $\sigma_{\mathbf{A}}$ is A (refer to Section 3). Thus, since $\sigma_{\mathbf{A}}$ is also a submonoid of $\mathbf{A}^*$, $\sigma_{\mathbf{A}}$ is a conucleus, and (ii) is proved.

(iii) We have for all $a, b, c, d \in \mathbf{A}$ with $b \leq a$ and $d \leq c$, $\sigma_{\mathbf{A}}((a \to^* b) \vee^* (c \to^* d)) = ac \to (cb \vee ad) = (ac \to cb) \vee (ac \to ad) = (a \to b) \vee (c \to d) = \sigma_{\mathbf{A}}(a \to^* b) \vee^* \sigma_{\mathbf{A}}(c \to^* d)$.
Thus σ_A is join-preserving. Moreover it follows from the proof of Lemma 8.3 that $\sigma_{\mathbf{A}}((a \to^* b) \wedge^* (c \to^* d))$ is the join of $t'_2(a, b, c, d) = t_{2,1}(a, c) \to t_{2,2}(a, b, c, d)$; $t'_3(a, c, d) = t_{3,1}(a, c) \to t_{3,2}(a, d)$; and $t'_4(a, b, c) = t_{4,1}(a, c) \to t_{4,2}(b, c)$. Also, by the definition of $\sigma_{\mathbf{A}}$, $\sigma_{\mathbf{A}}(a \to^* b) \wedge^* \sigma(c \to^* d) = (a \to b) \wedge (c \to d)$. We verify that these elements are equal. Let $i \in I$. If $a_i, c_i \neq 0_i$, then $t'_3(a, c, d)_i = t'_4(a, b, c)_i = 0_i$ and $t'_2(a, b, c, d)_i = (ac \to (cb \wedge ad))_i = ((a \to b) \wedge (c \to d))_i$. If $a_i = c_i = 0_i$, then $((a \to b) \wedge (c \to d))_i = t'_3(a, c, d)_i = 1_i$. If $a_i = 0_i$ and $c_i \neq 0_i$, then $((a \to b) \wedge (c \to d))_i = (c \to d)_i$, $t'_2(a, b, c, d)_i = t'_4(a, b, c)_i = 0_i$ and $t'_3(a, c, d) = (c \to d)_i$. The case where $c_i = 0_i$ and $a_i \neq 0_i$ is similar. $\square$

Definition 8.6. *Let $\mathcal{PA}_{cn}$ be the category with objects $\langle \mathbf{A}, \sigma \rangle$ consisting of a product algebra $\mathbf{A}$ augmented with a conucleus σ that is a lattice homomorphism and whose image generates $\mathbf{A}$. The morphisms of $\mathcal{PA}_{cn}$ are algebra homomorphisms (i.e., residuated lattice homomorphisms that preserve zero) that commute with the designated nuclei.*

We note that if $\langle \mathbf{A}, \sigma \rangle$ is an object of $\mathcal{PA}_{cn}$, then the image, A_σ, of σ is closed under multiplication and the lattice operations of $\mathbf{A}$. Moreover, in light of Lemma 3.1, it becomes a residuated lattice if the implication is defined by $x \to_\sigma y = \sigma(x \to y)$, for all $x, y \in A_\sigma$. It is actually a residuated bounded lattice, since $0 \in A_\sigma$. We shall denote this residuated bounded lattice by $\mathbf{A}_\sigma$.

Lemma 8.7. *If $\langle \mathbf{A}, \sigma \rangle$ is an object in $\mathcal{PA}_{cn}$, then $\mathbf{A}_\sigma$ is a* Π*MTL-algebra.*

Proof. As was noted above, $\mathbf{A}_\sigma$ is a residuated bounded lattice whose operations coincide with those of $\mathbf{A}$, except the implication which is given by $x \to_\sigma y = \sigma(x \to y)$, for all $x, y \in A_\sigma$. In what follows, we will write $\neg_\sigma x$ for $x \to_\sigma 0$.

It is clear that $\mathbf{A}_\sigma$ is a commutative, integral residuated bounded lattice. Moreover we have

$$(x \to_\sigma y) \vee (y \to_\sigma x) = \sigma(x \to y) \vee \sigma(y \to x) = \sigma(x \to y \vee y \to x) = 1,$$

and this equation in any commutative and integral residuated lattice implies representability.

Thus, in order to prove that $\mathbf{A}_\sigma$ is a ΠMTL-algebra, it remains to verify that $\neg_\sigma x \vee ((x \to_\sigma xy) \to_\sigma y) = 1$, for all $x, y \in \mathbf{A}_\sigma$. To begin with, note that for all $z, u \in \mathbf{A}$, $\sigma(z)\sigma(z \to u) \leq \sigma(z(z \to u)) \leq \sigma(u)$, and hence $\sigma(z \to u) \leq \sigma(z) \to \sigma(u)$. Thus if $x, y \in \mathbf{A}_\sigma$, then $\sigma(x \to xy) \to y \geq \sigma((x \to xy) \to y)$. This yields $(x \to_\sigma xy) \to_\sigma y = \sigma(\sigma(x \to xy) \to y) \geq \sigma((x \to xy) \to y)$. Since $\neg_\sigma x = \sigma(\neg x)$, we get successively $\neg_\sigma x \vee ((x \to_\sigma xy) \to_\sigma y) \geq \sigma(\neg x) \vee \sigma((x \to xy) \to y) = \sigma(\neg x \vee ((x \to xy) \to y)) = \sigma(1) = 1$. This concludes the proof. $\square$

Lemma 8.8. *Let $\langle \mathbf{B}, \sigma \rangle$ be an object of $\mathcal{PA}_{cn}$, let $\mathbf{B}^*_\sigma$ be the product algebra generated by $\mathbf{B}_\sigma$, and let $\sigma_{\mathbf{B}_\sigma}$ be the associated conucleus. Then $\langle \mathbf{B}, \sigma \rangle$ and $\langle \mathbf{B}^*_\sigma, \sigma_{\mathbf{B}_\sigma} \rangle$ are isomorphic objects of $\mathcal{PA}_{cn}$.*

Proof. Both $\mathbf{B}$ and $\mathbf{B}^*_\sigma$ are product algebras generated by $\mathbf{B}_\sigma$, therefore they are isomorphic as product algebras, by Lemma 8.4. Moreover the isomorphism Φ defined in the proof of Lemma 8.4 leaves the elements of $\mathbf{B}_\sigma$ fixed. Thus for every $x \in \mathbf{B}$, $\Phi(\sigma(x)) = \sigma(x)$. Now $\sigma(x)$ is the greatest element $z \in \mathbf{B}_\sigma$ such that $z \leq x$ in $\mathbf{B}$, and $\sigma_{\mathbf{B}_\sigma}(\Phi(x))$ is the greatest element $z \in \mathbf{B}_\sigma$ such that $z \leq \Phi(x)$ in $\mathbf{B}^*_\sigma$. Since Φ is an isomorphism of product algebras, we have, for all $z \in \mathbf{B}_\sigma$, $z \leq x$ iff $\Phi(z) = z \leq \Phi(x)$. Thus $\sigma_{\mathbf{B}_\sigma}(\Phi(x)) = \sigma(x) = \Phi(\sigma(x))$, and the claim is proved. $\square$

Recall that $\Pi\mathcal{MTL}$ is the category of ΠMTL-algebras and algebra homomorphisms.

Lemma 8.9. *For every morphism $\chi\colon \mathbf{A} \to \mathbf{B}$ in the category $\Pi\mathcal{MTL}$, let $\Pi(\chi)\colon \langle \mathbf{A}^*, \sigma_\mathbf{A} \rangle \to \langle \mathbf{B}^*, \sigma_\mathbf{B} \rangle$ be defined, for all elements $a, b \in \mathbf{A}$, by $\Pi(\chi)(a \to^*_A b) = \chi(a) \to^*_B \chi(b)$. Then $\Pi(\chi)$ is the unique $\mathcal{PA}_{cn}$-morphism from $\langle \mathbf{A}^*, \sigma_\mathbf{A} \rangle$ into $\langle \mathbf{B}^*, \sigma_\mathbf{B} \rangle$ extending χ.*

Proof. To begin with, note that $\Pi(\chi)$ is well defined. Indeed, if $a \to^*_A b = c \to^*_A d$, then $t(a,b,c,d) = 1$ holds in $\mathbf{A}$ (Lemma 8.3(b)). Hence $t(\chi(a),\chi(b),\chi(c),\chi(d)) = 1$ holds in $\mathbf{B}$, and thus, invoking Lemma 8.3(b) once again, we get that $\chi(a) \to^*_B \chi(b) = \chi(c) \to^*_B \chi(d)$.

We now verify that $\Pi(\chi)$ preserves the join operation. By the proof of Lemma 8.3(a), we have

$$\begin{aligned}\Pi(\chi)((a \to^*_A b) \vee^*_A (c \to^*_A d)) &= \Pi(\chi)(ac \to^*_A (bc \vee ad)) \\ &= \chi(ac) \to^*_B \chi(bc \vee ad) \\ &= \chi(a)\chi(c) \to^*_B (\chi(b)\chi(c) \vee \chi(a)\chi(d)) \\ &= (\chi(a) \to^*_B \chi(b)) \vee^*_B (\chi(c) \to^*_B \chi(d)) \\ &= \Pi(\chi)((a \to^*_A b) \vee^*_B \Pi(\chi)(c \to^*_A d)).\end{aligned}$$

We next prove that $\Pi(\chi)$ preserves multiplication. By the proof of Lemma 8.3 we have:

$$(a \to^*_A b) \cdot^*_A (c \to^*_A d) = t_1(a,b,c,d) \vee^*_A t_3(a,c,d) \vee^*_A t_4(a,b,c),$$

thus, since $\Pi(\chi)$ is compatible with join, $\Pi(\chi)((a \to^*_A b) \cdot^*_A (c \to^*_A d))$ reduces to $\Pi(\chi)(t_1(a,b,c,d)) \vee^*_B \Pi(\chi)(t_3(a,c,d)) \vee^*_B \Pi(\chi)(t_4(a,b,c))$.

On the other hand, $t_1(a,b,c,d) = t_{1,1}(a,c) \to^*_A t_{1,2}(a,b,c,d)$ where $t_{1,1}$ and $t_{1,2}$ are ΠMTL-algebra terms. Thus

$$\Pi(\chi)t_1(a,b,c,d) = t_{1,1}(\chi(a),\chi(c)) \to^*_B t_{1,2}(\chi(a),\chi(b),\chi(c),\chi(d)).$$

Similarly, we obtain

$$\Pi(\chi)t_3(a,c,d) = t_{3,1}(\chi(a),\chi(c)) \to^*_B t_{3,2}(\chi(c),\chi(d)),$$

$$\Pi(\chi)t_4(a,b,c) = t_{4,1}(\chi(a),\chi(c)) \to^*_B t_{4,2}(\chi(b),\chi(c)).$$

And,

$$\Pi(\chi)(a \to^*_A b) \cdot^*_B \Pi(\chi)(c \to^*_A d)) = (\chi(a) \to^*_B \chi(b)) \cdot^*_B (\chi(c) \to^*_B \chi(d)).$$

Therefore, by the proof of Lemma 8.3, $\Pi(\chi)(a \to^*_A b) \cdot^*_B \Pi(\chi)(c \to^*_A d)$ is the join of the following: $t_{1,1}(\chi(a),\chi(c)) \to^*_B t_{12}(\chi(a),\chi(b),\chi(c),\chi(d))$, $t_{31}(\chi(a),\chi(c)) \to^*_B t_{3,2}(\chi(c),\chi(d))$ and $t_{4,1}(\chi(a),\chi(c)) \to^*_B t_{4,2}(\chi(b),\chi(c))$. It follows:

$$\Pi(\chi)((a \to^*_A b) \cdot^*_A (c \to^*_A d)) = \Pi(\chi)(a \to^*_A b) \cdot^*_B \Pi(\chi)(c \to^*_A d).$$

One can verify in a quite analogous manner that $\Pi(\chi)$ preserves meet and implication.

We now prove that $\Pi(\chi)$ commutes with the conuclei. We have $\Pi(\chi)(\sigma_{\mathbf{A}}(a \to^*_A b)) = \Pi(\chi)(a \to b) = \chi(a \to b) = \chi(a) \to \chi(b)$. On the other hand, $\sigma_{\mathbf{B}}(\Pi(\chi)(a \to^*_A b)) = \sigma_{\mathbf{B}}(\chi(a) \to^*_B \chi(b)) = \chi(a) \to \chi(b)$, and the claim is proved.

Lastly, it is clear that any homomorphism from $\langle \mathbf{A}^*, \sigma_{\mathbf{A}} \rangle$ to $\langle \mathbf{B}^*, \sigma_{\mathbf{B}} \rangle$ extending χ must coincide with $\Pi(\chi)$. □

We now define explicitly a pair of functors that will establish the equivalence of the categories $\mathit{\Pi}\mathcal{MTL}$ and $\mathcal{PA}_{cn}$.

Definition 8.10.

(i) *For every object* $\mathbf{A}$ *in* $\mathit{\Pi}\mathcal{MTL}$, *let* $\Pi(\mathbf{A}) = \langle \mathbf{A}^*, \sigma_{\mathbf{A}} \rangle$.

(ii) *For any* $\mathit{\Pi}\mathcal{MTL}$*-morphism* $\chi : \mathbf{A} \to \mathbf{B}$, *let* $\Pi(\chi)$ *be the morphism* $\Pi(\chi) : \langle \mathbf{A}^*, \sigma_{\mathbf{A}} \rangle \to \langle \mathbf{B}^*, \sigma_{\mathbf{B}} \rangle$ *defined by* $\Pi(\chi)(a \to^*_{\mathbf{A}^*} b) = \chi(a) \to^*_{\mathbf{B}^*} \chi(b)$.

(iii) *For every object* $\langle \mathbf{M}, \sigma \rangle$ *in* $\mathcal{PA}_{cn}$, *let* $\Pi^{-1}\langle \mathbf{M}, \sigma \rangle = \mathbf{M}_\sigma$.

(iv) *For every* $\mathcal{PA}_{cn}$*-morphism* $\varphi : \langle \mathbf{M}, \sigma_{\mathbf{M}} \rangle \to \langle \mathbf{N}, \sigma_{\mathbf{N}} \rangle$, *let* $\Pi^{-1}(\varphi) : \mathbf{M}_{\sigma_{\mathbf{M}}} \to \mathbf{N}_{\sigma_{\mathbf{N}}}$ *denote the restriction of* φ *on* $\mathbf{M}_{\sigma_{\mathbf{M}}}$.

Theorem 8.11. *The pair of functors* Π *and* Π^{-1} *constitute an equivalence of the categories* $\mathit{\Pi}\mathcal{MTL}$ *and* $\mathcal{PA}_{cn}$.

Proof. As in the proof of Theorem 4.7, it is sufficient to prove that Π is full and faithful, and that for every object $\langle \mathbf{A}, \sigma \rangle$ of $\mathcal{PA}_{cn}$, $\mathbf{A}$ and $\Pi(\Pi^{-1}(\mathbf{A}))$ are isomorphic. For any two objects $\mathbf{A}$, $\mathbf{B}$ of $\mathit{\Pi}\mathcal{MTL}$ and for any two morphisms $\phi, \psi \in \mathit{Hom}(\mathbf{A}, \mathbf{B})$, if $\phi \neq \psi$, then $\Pi(\phi) \neq \Pi(\psi)$, as $\Pi(\phi)$ extends ϕ and $\Pi(\psi)$ extends ψ. Thus Π is faithful.

Now let $\gamma \in \mathit{Hom}(\Pi(\mathbf{A}), \Pi(\mathbf{B}))$. Then its restriction $\Pi^{-1}(\gamma)$ to $\mathbf{A}$ is a morphism from $\mathbf{A}$ into $\mathbf{B}$, and by Lemma 8.9, has a unique extension to a morphism from $\Pi(\mathbf{A})$ to $\Pi(\mathbf{B})$. Now both $\Pi(\Pi^{-1}(\gamma))$ and γ are such morphisms, and hence they must coincide. We have verified that Π is full.

Lastly, Lemma 8.8 implies that if $\langle \mathbf{B}, \sigma \rangle$ is an object in $\mathcal{PA}_{cn}$, $\langle \mathbf{B}, \sigma \rangle$ and $\Pi(\Pi^{-1}\langle \mathbf{B}, \sigma \rangle)$ are isomorphic. The proof of the theorem is now complete. □

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