GÖDEL INCOMPLETENESS IN AF C*-ALGEBRAS

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ABSTRACT. For any (possibly, non-unital) AF C*-algebra A with comparability of projections, let $D(A)$ be the Elliott partial monoid of A, and $G(A)$ the dimension group of A with scale $D(A)$. For $D \subseteq D(A)$ a generating set of $G(A)$ let $\mathcal{P}$ be the set of all formal inequalities $a_1 + \cdots + a_k \leq b_1 + \cdots + b_l$ satisfied by $G(A)$, for any $a_i, b_j \in D$. By Elliott's classification, $\mathcal{P}$ together with the list of all sums $a_1 + \cdots + a_k \in D(A)$ uniquely determines A. Can $\mathcal{P}$ be Gödel incomplete, i.e., effectively enumerable but undecidable? We give a negative answer in case D is finite, and a positive answer in the general case. We also show that the range of the map $A \mapsto D(A)$ precisely consists of all countable partial abelian monoids satisfying the following three conditions: (i) $a + b = a + c \Rightarrow b = c$, (ii) $a + b = 0 \Rightarrow a = b = 0$ and (iii) $\forall a, b \in E \;\; \exists c \in E$ such that either $a + c = b$ or $b + c = a$.

1. INTRODUCTION

We assume familiarity with Elliott's classification [7, 6]. An *AF C*-algebra* A is the inductive limit of a sequence of finite-dimensional C*-algebras. A is not assumed to have a unit. For every AF C*-algebra A let $D(A)$ be the countable set of (Murray-von Neumann) equivalence classes of projections in A. The operation of adding two orthogonal projections equips $D(A)$ with a partial addition $+$ making $D(A)$ into a partial abelian monoid. Elliott [7, 4.3 and Section 7] proved that for any two AF C*-algebras A and B, $D(A)$ is isomorphic to $D(B)$ if and only if A is isomorphic to B. Every $D(A)$ arises as a *scale*, i.e., a directed hereditary (=order-convex) generating subset of the positive cone of $G(A)$, of a unique countable scaled dimension group $G(A)$.

While Bratteli diagrams provide useful *infinite* combinatorial presentations of all AF C*-algebras, as an effect of Elliott's classification, many interesting AF C*-algebras can also be presented as *finite* strings of symbols: this is a prerequisite for the study of (effective) enumerability and decidability problems for classes of AF C*-algebras, once they are presented by integer matrices [7, 8, 3, 4], propositions in infinite-valued Lukasiewicz logic [13], or abstract simplicial complexes [14]. See [12] for background on enumerability and decidability.

Following a time-honored tradition, in this paper we shall present every AF C*-algebra A by choosing a suitable set $D \subseteq D(A)$ of *generators* of $G(A)$ and then listing the *relations* (i.e., the inequalities between formal sums of generators) satisfied by $G(A)$. In order to uniquely recover A, all sums $a_1 + \cdots + a_m$ existing in $D(A)$ must also be listed. There is no need to consider the order of $G(A)$, since $x \geq y$ in $G(A)$ iff $x = y + z$ for some $z = b_1 + \cdots + b_n, \;\; (b_i \in D(A))$.

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An AF C*-algebra A is said to have *comparability* (of projections in the sense of Murray-von Neumann) if for any two projections in A, one of them is the support of a partial isometry whose range is contained in the other. This condition is equivalent to the order of $D(A)$ (equivalently, the order of $G(A)$) being total. Notable examples of AF C*-algebras with comparability include (i) the Effros-Shen C*-algebras $\mathfrak{F}_\theta$, for all $\theta \in \mathbb{R} \setminus \mathbb{Q}$ [6, p.65], (ii) the C*-algebra $\mathfrak{K}$ of compact operators, (iii) all Behnke-Leptin separable postliminary C*-algebras with finitely many closed two-sided ideals $J_1, \ldots, J_k$, having the additional property that $J_i \supseteq J_{i+1}$ for each $i = 1, \ldots, k-1$ (see [2] and [8, 5.2]), and (iv) Glimm UHF algebras [6, p.28]. The scaled dimension group of any AF C*-algebra A of type (i)-(iii) has a finite set $D \subseteq D(A)$ of generators. Identifying D with a set of symbols, we denote by D^* the set of words (=tuples) of D. For any $\mathbf{a} = (a_1, \ldots, a_k) \in D^*$ we let $\sum \mathbf{a}$ be short for $a_1 + \cdots + a_k$

Our main result is the following generalization of [13, 6.1]:

Theorem 1.1. *Let A be an AF C*-algebra with comparability, and suppose $D \subseteq D(A)$ is a finite generating set of $G(A)$. Suppose the set*

$$\mathcal{P} = \{(\mathbf{a}, \mathbf{b}) \in D^* \times D^* \mid G(A) \text{ satisfies } \sum \mathbf{a} \leq \sum \mathbf{b}\}$$

is enumerable. Then $\mathcal{P}$ is decidable.

The finiteness assumption for D cannot be dropped (Theorem 2.3).

While the constructions [7], [6, p.46] of $G(A)$ from $D(A)$ refer to A, in Section 3, following Baer [1] we shall give an intrinsic A-free definition of $G(A)$ from $D(A)$. As a preliminary step, the class of Elliott's partial monoids of AF C*-algebras with comparability will be characterized as follows: [1]

Theorem 1.2. *Let $E = (E, 0, +)$ be a countable partial abelian monoid. Then the following conditions are equivalent:*
(i) E is the Elliott partial monoid of some AF C-algebra with comparability.*
(ii) E satisfies the conditions
Cancellativity: $a + b = a + c \Rightarrow b = c$;
Positivity: $a + b = 0 \Rightarrow a = b = 0$;
Comparability: $\forall a, b \in E \;\; \exists c \in E$ *such that either* $a + c = b$ *or* $b + c = a$.

For any countable partial monoid E satisfying these three conditions, in Proposition 3.3 the AF C*-algebra A with $D(A) \cong E$ is easily obtained, generalizing Elliott's ultrasimplicial construction [8, p.44].

Sections 2 and 3 can be read independently.

2. Proof of Theorem 1.1

We need some elementary facts about complexes of convex polyhedral cones in $\mathbb{R}^n$, and abelian lattice-ordered groups (for short, ℓ-groups). We briefly summarize the notions and notations used in this section.

Rational simplicial cones and nonsingular fans [9, 10]. A *fan* is a finite complex Δ of rational simplicial cones in $\mathbb{R}^n$. For each $i = 1, \ldots, n$ we let $\Delta^{(i)}$ denote the set of i-dimensional cones in Δ. In particular, we write

$$\Delta^{(1)} = \{\rho_1, \ldots, \rho_k\} = \{\mathbb{R}_{\geq 0}\mathbf{v}_1, \ldots, \mathbb{R}_{\geq 0}\mathbf{v}_k\} = \{\langle \mathbf{v}_1 \rangle, \ldots, \langle \mathbf{v}_k \rangle\},$$

[1] compare with with Pulmannová's result in [16],[5, 3.3.11] for unital AF C*-algebras.

for the set of one-dimensional cones (=rays) of Δ, and we say that the ray ρ_i of Δ is *generated by the primitive* [2] *vector* $\mathbf{v}_i \in \mathbb{Z}^n$. The latter is uniquely determined. We shall similarly express every d-dimensional cone

$$\sigma = \langle \mathbf{v}_1, \ldots, \mathbf{v}_d \rangle = \mathbb{R}_{\geq 0}\mathbf{v}_1 + \cdots + \mathbb{R}_{\geq 0}\mathbf{v}_k$$

as the positive real span of its primitive generators $\mathbf{v}_1, \ldots, \mathbf{v}_d \in \mathbb{Z}^n$. The only 0-dimensional cone is the singleton $\{0\}$. A fan Δ is *complete* if $\bigcup\{\sigma : \sigma \in \Delta\} = \mathbb{R}^n$. If Σ and Δ are complete fans and every cone of Δ is contained in some cone of Σ, we say that Δ is a *subdivision* of Σ, and we write $\Delta \leq \Sigma$. Every fan Σ can be effectively presented by exhibiting the list of its cones, each cone being presented by the list of its primitive generating vectors. In particular, for a complete fan Σ it is sufficient to list the cones in $\Sigma^{(n)}$. A fan Δ is said to be *nonsingular* (*regular* in [9]) if every cone of Δ is generated by a part of a basis of $\mathbb{Z}^n$.

Free ℓ-groups, [11]. An *ℓ-group* is an algebra $(G, +, -, 0, \vee, \wedge)$ such that $(G, +, -, 0)$ is an abelian group, $(G, \vee, \wedge)$ is a lattice, and $x + (y \vee z) = (x + y) \vee (x + z)$ for all $x, y, z \in G$. Since ℓ-groups form an equational class, every finite set $D = \{g_1, \ldots, g_n\}$ is a *free generating set* of a certain ℓ-group, denoted $\mathcal{A}_n$. In other words, [11, p.87], $g_1, \ldots, g_n$ generate $\mathcal{A}_n$, and for any ℓ-group G and map $\varphi \colon \{g_1, \ldots, g_n\} \to G$, φ can be uniquely extended to an ℓ-homomorphism of $\mathcal{A}_n$ into G. $\mathcal{A}_n$ is uniquely determined up to ℓ-isomorphism, and is known as *the free n-generator ℓ-group.*

For Y a topological space let $C(Y, \mathbb{R})$ denote the ℓ-group of all real-valued continuous functions defined on Y, with the pointwise ℓ-group operations $+, -, \vee, \wedge$ of $\mathbb{R}$. Here, $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$. For each $i = 1, \ldots, n$ let $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$ be the ith projection function, $\pi_i(x_1, \ldots, x_n) = x_i$. As is well known, [11, 5.A], $\mathcal{A}_n$ is the ℓ-subgroup of $C(\mathbb{R}^n, \mathbb{R})$ generated by $\pi_1, \ldots, \pi_n$, and $\{\pi_1, \ldots, \pi_n\}$ is a *free* generating set of $\mathcal{A}_n$. One easily sees that each element of $\mathcal{A}_n$ is a real-valued continuous piecewise linear homogeneous function f defined on $\mathbb{R}^n$, and each linear piece of f has integer coefficients.

Let $X_1, \ldots, X_n$ be distinct symbols, called *variables.* Let τ be an ℓ-group term. By writing $\tau = \tau(X_1, \ldots, X_n)$ we mean that the variables occurring in τ are all members of the set $\{X_1, \ldots, X_n\}$. For any ℓ-group L and elements $l_1, \ldots, l_n \in L$, the map $X_i \mapsto l_i$ uniquely extends in the usual way to an interpretation of each term $\tau(X_1, \ldots, X_n)$ as an element of L, denoted $\tau(l_1, \ldots, l_n)$. In particular, the map $X_i \mapsto \pi_i$ extends to an interpretation $\tau \mapsto f_\tau$ of each term $\tau = \tau(X_1, \ldots, X_n)$ as a function of $\mathcal{A}_n$, which for short we denote by f_τ instead of $\tau(\pi_1, \ldots, \pi_n)$. We also say that τ *represents* f_τ.

Lemma 2.1. *For every ℓ-group term $\tau = \tau(X_1, \ldots, X_n)$, let $Zf_\tau = \{\mathbf{x} \in \mathbb{R}^n \mid f_\tau(\mathbf{x}) = 0\}$ be the zeroset of f_τ.*

(1) *Then Zf_τ is a finite union of rational polyhedral cones, and the map $\tau \mapsto Zf_\tau$ is effectively computable.*
(2) *There is an effective procedure which, over input τ outputs a complete nonsingular fan Σ_τ with $\Sigma_\tau^{(n)} = \{\sigma_1, \ldots, \sigma_q\}$, together with a tuple $l_1, \ldots, l_q$ of linear homogeneous polynomials with integer coefficients such that $f_\tau = l_i$ on σ_i, for each $1 \leq i \leq q$.*

Proof. (1) is an immediate consequence of $f_\tau : \mathbb{R}^n \to \mathbb{R}$ being continuous piecewise linear, each piece with integer coefficients.

[2] in the sense that the greatest common divisor of the coordinates of $\mathbf{v}_i$ is equal to 1.

To prove (2), we first write f_τ in normal form (as in the proof of [11, 5.A])

$$f_\tau = \bigvee_{1\leq r\leq t} \bigwedge_{1\leq s_r\leq m_r} l_{rs_r} \tag{1}$$

where every l_{rs_r} is a linear homogeneous function with integer coefficients. Let $l_1,\ldots,l_k$ display the set of all the l_{rs_r}'s. For every permutation μ of $1,\ldots,k$, the set of all solutions $\mathbf{x}\in\mathbb{R}^n$ of the system of inequalities

$$l_{\mu(1)}(\mathbf{x}) \leq l_{\mu(2)}(\mathbf{x}) \leq \cdots \leq l_{\mu(k)}(\mathbf{x})$$

is a rational polyhedral cone σ_μ. We take the complete fan Θ to be the set of all faces of all σ_μ's. A set of vertices for each cone in Θ can be effectively computed, and for every n-dimensional cone $\sigma\in\Theta$ we can effectively choose an element among $l_1,\ldots,l_k$ that agrees with f_τ on σ. Using the well known desingularization procedure [10, p.48], [9, VI.8.5] we now effectively construct a complete nonsingular fan Σ_τ which is a subdivision of Θ, and we attach to each cone $\sigma\in\Sigma_\tau^{(n)}$ a linear homogeneous function $l_\sigma\in\{l_1,\ldots,l_k\}$ with integer coefficients such that $f_\tau = l_\sigma$ on σ. □

Lemma 2.2. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{A}_n$, and let $T_\mathfrak{p}$ be the set of ℓ-group terms $\tau_1,\tau_2,\ldots$ representing the elements $f_1,f_2,\ldots$ of $\mathfrak{p}$. Suppose $T_\mathfrak{p}$ is enumerable. For each $j=1,2,\ldots$ define the zeroset $Z_j = Z(|f_1|+\cdots+|f_j|)$. We then have*

(i) *For every complete nonsingular fan Δ there exists an index i and a cone $\sigma\in\Delta^{(n)}$ such that $Z_i\subseteq\sigma$.*
(ii) *For every $g\in\mathcal{A}_n$ $g\in\mathfrak{p}$ iff there exists i such that $Z_i\subseteq Zg$.*

Proof. (i) By Lemma 2.1 and our hypothesis about $T_\mathfrak{p}$, there is an effective procedure which, over input j outputs a finite set of rational polyhedral cones whose union coincides with Z_j. Further, given any rational polyhedral cone σ, it is decidable whether Z_i is a subset of σ. Given the nonsingular fan Δ, the nonsingularity of Δ is decidable. For every n-dimensional cone $\gamma\in\Delta$, let $\upsilon_\gamma\colon\mathbb{R}^n\to\mathbb{R}$ be the only piecewise linear homogeneous function which vanishes on γ and takes value 1 on all primitive generating vectors of $\Delta^{(1)}$, other than those of γ. From the assumed nonsingularity of Δ it follows that every linear piece of υ_γ has integer coefficients, whence $\upsilon_\gamma\in\mathcal{A}_n$ for each $\gamma\in\Delta^{(n)}$. By construction, $\bigwedge\{\upsilon_\gamma \mid \gamma\in\Delta^{(n)}\} = 0\in\mathfrak{p}$; since $\mathfrak{p}$ is prime there exists $\sigma\in\Delta^{(n)}$ with $\upsilon_\sigma\in\mathfrak{p}$. (In general, σ is not unique.) It follows that for some integer $i>0$, some positive integer multiple of the function $|f_1|+\cdots+|f_i|$ dominates υ_σ, whence in particular, $Z_i\subseteq Z(\upsilon_\sigma)=\sigma$.

(ii) One direction is trivial. If $Z_i\subseteq Zg$, then $Z(|f_1|+\cdots+|f_i|)\subseteq Z|g|$, and a routine compactness argument shows that $|g|\leq m(|f_1|+\cdots+|f_i|)$, for some integer $m\geq 1$. Since $|f_1|+\cdots+|f_i|\in\mathfrak{p}$, both $|g|$ and g are in $\mathfrak{p}$. □

For any $g\in\mathcal{A}_n$ and $\mathbf{u}\in\mathbb{R}^n$, we define $\mathrm{D}_\mathbf{u}g\colon\mathbb{R}^n\to\mathbb{R}$ of g at $\mathbf{u}$ by

$$\mathrm{D}_\mathbf{u}g(\mathbf{v}) = \lim_{\epsilon\to 0^+}\frac{g(\mathbf{u}+\epsilon\mathbf{v})-g(\mathbf{u})}{\epsilon} \qquad (\forall\ \mathbf{v}\in\mathbb{R}^n).$$

For any fixed $\mathbf{w}\in\mathbb{R}^n$ we have $\mathrm{D}_\mathbf{w}g\in\mathcal{A}_n$. We further let

$$[[\mathbf{w}]] = (\mathbf{w}^\perp\cap\mathbb{Z}^n)^\perp.$$

Equivalently, $[[\mathbf{w}]]$ is the intersection of all subspaces of $\mathbb{R}^n$ that contain $\mathbf{w}$ and are definable via linear equations with integer coefficients. An orthonormal tuple

$\vec{\mathbf{u}} = (\mathbf{u}_1, \dots, \mathbf{u}_t)$ of elements of $\mathbb{R}^n$ is said to be $\mathbb{Z}$-*reduced* if, for every $1 \leq i < t$, we have $\mathbf{u}_{i+1} \in ([[\mathbf{u}_1]] + \cdots + [[\mathbf{u}_i]])^{\perp}$. For any $\mathbb{Z}$-reduced tuple $\vec{\mathbf{u}}$ we further let

$$(2) \qquad \mathfrak{p}_{\vec{\mathbf{u}}} = \{f \in \mathcal{A}_n \mid \mathrm{D}_{\mathbf{u}_t}\mathrm{D}_{\mathbf{u}_{t-1}} \dots \mathrm{D}_{\mathbf{u}_1} f = 0 \text{ on } [[\mathbf{u}_1]] + \cdots + [[\mathbf{u}_t]]\}.$$

By [15, 4.8],

$$(3) \qquad \mathfrak{p} = \mathfrak{p}_{\vec{\mathbf{u}}} \text{ for a unique } \mathbb{Z}\text{-reduced tuple } \vec{\mathbf{u}} = (\mathbf{u}_1, \dots, \mathbf{u}_t)$$

By definition, the linear space $[[\mathbf{u}_1]] + \cdots + [[\mathbf{u}_t]]$ has a basis over $\mathbb{R}$ consisting of vectors $\mathbf{v}_1, \dots, \mathbf{v}_r$ in $\mathbb{Z}^n$, which we assume to be given.

End of proof of Theorem 1.1**.** Write $D = \{d_1, \dots, d_n\}$. Since the AF C*-algebra A has comparability and D is finite, its dimension group $G = G(A)$ is a totally ordered abelian group with n generators. The map $\pi_i \mapsto d_i$ uniquely extends to an ℓ-homomorphism $\theta\colon \mathcal{A}_n \to G$. The map θ is onto G, because D generates G. The elementary theory of ℓ-groups shows that the kernel of θ is a prime ideal $\mathfrak{p}$ of $\mathcal{A}_n$. Further, for every ℓ-group term $\tau = \tau(X_1, \dots, X_n)$ we have

$$(4) \qquad \theta\colon \tau(\pi_1, \dots, \pi_n) \mapsto \tau(d_1, \dots, d_n)$$

and

$$(5) \qquad \mathcal{A}_n \models \tau(\pi_1, \dots, \pi_n) \in \mathfrak{p} \Leftrightarrow G \models \tau(d_1, \dots, d_n) = 0.$$

By hypothesis, we can effectively enumerate all pairs $(\mathbf{a}, \mathbf{b}) \in D^* \times D^*$ such that $(\sum \mathbf{a} - \sum \mathbf{b}) \vee 0$ is the zero element of G. Writing $\mathbf{a} = (a_1, \dots, a_p) = (d_{i(1)}, \dots, d_{i(p)})$ and $\mathbf{b} = (b_1, \dots, b_q) = (d_{j(1)}, \dots, d_{j(q)})$ and replacing every d_i by X_i in $(\sum \mathbf{a} - \sum \mathbf{b}) \vee 0$, we obtain an ℓ-group term

$$\eta_{\mathbf{ab}} = \eta_{\mathbf{ab}}(X_1, \dots, X_n) = (\sum_{i=1}^{p} X_{i(t)} - \sum_{j=1}^{q} X_{j(t)}) \vee 0,$$

and we have

$$(6) \qquad (\mathbf{a}, \mathbf{b}) \in \mathcal{P} \Leftrightarrow G \models \eta_{\mathbf{ab}}(d_1, \dots, d_n) = 0 \Leftrightarrow \mathcal{A}_n \models \eta_{\mathbf{ab}}(\pi_1, \dots, \pi_n) \in \mathfrak{p}.$$

Claim. The set $T_{\mathfrak{p}}$ of ℓ-group terms $\tau = \tau(X_1, \dots, X_n)$ such that f_τ belongs to $\mathfrak{p}$ is enumerable.

As a matter of fact, by (4)-(5), using the abbreviation $\mathbf{d} = (d_1, \dots, d_n)$ we have

$$(7) \qquad \mathcal{A}_n \models \tau(\pi_1, \dots, \pi_n) \in \mathfrak{p} \Leftrightarrow G \models \theta(\tau(\pi_1, \dots, \pi_n)) = 0 \Leftrightarrow G \models \tau(\mathbf{d}) = 0.$$

Now for some $\mathbf{a}$ and $\mathbf{b}$ in D^*, $\tau(d_1, \dots, d_n)$ can be effectively rewritten in the *purely additive* form $\tau^*(d_1, \dots, d_n) = \sum \mathbf{a} - \sum \mathbf{b}$, in the sense that $G \models \tau(d_1, \dots, d_n) = \sum \mathbf{a} - \sum \mathbf{b}$. This can be easily proved by induction on the number of lattice operations $\vee, \wedge$ in τ. The basis is trivial, For the induction step, suppose $\tau = \sigma_1 \vee \sigma_2$. By induction, $\sigma_1(d_1, \dots, d_n)$ and $\sigma_2(d_1, \dots, d_n)$ have already been given purely additive forms as $\sum \mathbf{a}_1 - \sum \mathbf{b}_1$ and $\sum \mathbf{a}_2 - \sum \mathbf{b}_2$, for some $\mathbf{a}_1, \mathbf{b}_1, \mathbf{a}_2, \mathbf{b}_2 \in D^*$. Since $\mathcal{P}$ is enumerable and G is totally ordered, for some effectively given $i \in \{1, 2\}$ we can replace $\tau(d_1, \dots, d_n)$ by the purely additive form $\sum \mathbf{a}_i - \sum \mathbf{b}_i$ of the element $e = \max\{\sigma_1(d_1, \dots, d_n), \sigma_2(d_1, \dots, d_n)\}$. This yields the promised rewriting of $\tau(d_1, \dots, d_n)$ as $\sum \mathbf{a} - \sum \mathbf{b}$.

Recalling (7) we can write

$$\mathcal{A}_n \models \tau(\pi_1, \dots, \pi_n) \in \mathfrak{p} \Leftrightarrow G \models \sum \mathbf{a} - \sum \mathbf{b} = 0.$$

Trivially, the set of pairs of tuples $(\mathbf{a},\mathbf{b}) \in D^* \times D^*$ with both $(\mathbf{a},\mathbf{b})$ and $(\mathbf{b},\mathbf{a})$ members of $\mathcal{P}$ is enumerable. Our claim is settled.

To conclude the proof, in the light of (6), it is enough to give an algorithm which, over any input $(\mathbf{a},\mathbf{b})$, decides if the function $\eta_{\mathbf{ab}}(\pi_1,\ldots,\pi_n)$ belongs to $\mathfrak{p}$. Observe that the map $(\mathbf{a},\mathbf{b}) \mapsto \eta_{\mathbf{ab}}(X_1,\ldots,X_n)$ is effectively computable, and write for short $f_{\mathbf{ab}}$ instead of $\eta_{\mathbf{ab}}(\pi_1,\ldots,\pi_n)$. Using Lemma 2.1 construct a nonsingular complete fan $\Sigma = \Sigma_{\eta_{\mathbf{ab}}}$ with $\Sigma^{(n)} = \{\sigma_1,\ldots,\sigma_q\}$, and the q-tuple of linear homogeneous polynomials with integer coefficients $l_1,\ldots,l_q$, such that $f_{\mathbf{ab}} = l_i$ on each σ_i. [3] Since by our claim, $T_{\mathfrak{p}}$ is enumerable, let us display the the zerosets $Z_1 \supseteq Z_2 \supseteq \ldots$ in the enumeration of Lemma 2.2(i), and wait for the first pair of integers $i = 1,2,3,\ldots$ and $j \in \{1,\ldots,q\}$ yielding a zeroset Z_i and a cone $\sigma_j \in \Sigma^{(n)}$ such that $Z_i \subseteq \sigma_j$. It follows that

$$(8) \qquad f_{\mathbf{ab}} \in \mathfrak{p} \quad \Leftrightarrow \quad l_j(\mathbf{v}_1) = \cdots = l_j(\mathbf{v}_r) = 0.$$

Indeed, from $f_{\mathbf{ab}} - l_j = 0$ on $\sigma_j \subseteq Z_i$ we get $f_{\mathbf{ab}} - l_j \in \mathfrak{p}$, by Lemma 2.2(ii). Therefore, $f_{\mathbf{ab}} \in \mathfrak{p} \Leftrightarrow l_j \in \mathfrak{p}$. The linearity of l_j ensures $\mathrm{D}_{\mathbf{w}} f = f$ for every $\mathbf{w} \in \mathbb{R}^n$. From (2)-(3) it follows that $l_j \in \mathfrak{p} \Leftrightarrow l_j \in \mathfrak{p}_{\vec{\mathbf{u}}} \Leftrightarrow l_j = 0$ on $[[\mathbf{u}_1]] + \cdots + [[\mathbf{u}_t]] \Leftrightarrow l_j(\mathbf{v}_1) = \cdots = l_j(\mathbf{v}_r) = 0$, as required to settle (8). Since the tuple $\mathbf{v}_1,\ldots,\mathbf{v}_r$ in $\mathbb{Z}^n$ is given, the computations of the integers $l_j(\mathbf{v}_1) \cdots = l_j(\mathbf{v}_r)$ can be done effectively. Lemmas 2.1-2.2 ensures that all constructions are effective, whence the above procedure decides whether $(\mathbf{a},\mathbf{b}) \in \mathcal{P}$.

The proof of Theorem 1.1 is complete.

Remarks. 1. In general, the tuple of integer vectors $\mathbf{v}_1,\ldots,\mathbf{v}_r$ is *not* effectively computable from a computer program I enumerating $\mathcal{P}$. Thus our decidability result is highly nonuniform: there is no mechanical procedure upgrading I to a computer program I^+ deciding $\mathcal{P}$.
2. The case when $\mathfrak{p}$ is maximal was dealt with in [13, 6.1].

Dropping the finiteness assumption. Given an AF C*-algebra A with comparability, suppose $D \subseteq D(A)$ to be an infinite generating set for $G(A)$. In this section we shall see that Theorem 1.1 no longer holds. There may be various reasons for the Gödel incompleteness of a presentation of A by generators and relations of $G(A)$.

First of all, the identification in Theorem 1.1 of $D \subseteq D(A) \subseteq G(A)$ with a set of symbols in a finite alphabet is no longer possible: For certain A, no matter how "reasonably" we code each $d \in D$ as a word $w(d)$ over a finite alphabet Ω, the Gödel incompleteness of the resulting presentation of A is simply due to the fact that the range of the function w is an undecidable subset of the set Ω^* of words over Ω.

For instance, let $R = \{1/2, 1/3, 1/5, \ldots\}$ be the set of reciprocals of prime numbers. Let $D \subseteq R$ be a Gödel incomplete subset of R. Let G be the subgroup of $\mathbb{Q}$ generated by the elements of D. Code the elements of D in the usual way, using the binary notation. Let A be the UHF-algebra given by $G(A) = G$ with the stable scale G^+. Let $\mathcal{P}$ be the set of all pairs of tuples of words coding those pairs $((a_1,\ldots,a_k),(b_1,\ldots,b_l)) \in D^k \times D^l$ such that $G(A) \models a_1 + \cdots + a_k \leq b_1 + \cdots + b_l\}$. Then $\mathcal{P}$ is Gödel incomplete.

[3]direct inspection on $f_{\mathbf{ab}}$ shows that one can always assume that there are at most two distinct elements among the l_i, and one of them is the zero function.

There may be deeper reasons for the Gödel incompleteness of a presentation of an AF C*-algebra A with comparability:

Theorem 2.3. *There exists an AF C*-algebra A with comparability, together with a generating set $D \subseteq D(A)$ of $G(A)$, and a one-one map $^{-}$ of D onto $N = \{\text{I}, \text{II}, \text{III}, \ldots\}$ such that the set*

$$\mathcal{P} = \{((\overline{a_1}, \ldots, \overline{a_m}), (\overline{b_1}, \ldots, \overline{b_n})) \in N^m \times N^n \mid G(A) \models a_1 + \cdots + a_m \leq b_1 + \cdots + b_n\}$$

is enumerable and undecidable.

Proof. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let P be an enumerable undecidable subset of odd prime numbers. Let $\mathcal{G}$ denote the group of all polynomials $p(x) = a_1 x^{m_1} + \cdots + a_t x^{m_t}$ with exponents $m_1 > \cdots > m_t$ in the complementary set $\mathbb{N} \setminus P$, and integer coefficients, defined on the halfline $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. As an abelian group, $\mathcal{G}$ is freely generated by the set $\mathcal{M}$ of monomials x^m, for $m \in \mathbb{N} \setminus P$. For $p \in \mathcal{G}$ a nonzero polynomial, the first nonzero coefficient a_i in p is said to be the *leading coefficient* of p. For any two polynomials $p, q \in \mathcal{G}$ we write $p \prec q$ iff $p(x) < q(x)$ for all suitably large $x \in \mathbb{R}^+$. The binary relation $\preceq$ endows $\mathcal{G}$ with the structure of a totally ordered abelian group, also denoted $\mathcal{G}$. The positive cone $\mathcal{G}^+$ is given by the zero polynomial together with those polynomials whose leading coefficient is > 0. Let A be the AF C*-algebra with comparability, whose scaled dimension group $G(A)$ coincides with $\mathcal{G}$ equipped with the stable scale $D(A) = \mathcal{G}^+$. As a generating set $D \subseteq D(A)$ for $G(A)$ let us choose the set

$$D = \{x^k \mid k \in \mathbb{N} \setminus P\} \cup \{2x^{k+1} \mid k \in P\}.$$

Observe that D is redundant: indeed, from $k \in P \Rightarrow k+1 \in \mathbb{N} \setminus P$ it follows that the subset of D given by $\{x^k \mid k \in \mathbb{N} \setminus P\}$ already generates G. Taking advantage of this redundancy, we define the one-one map ι from N onto D by

$$\iota(\underbrace{\text{III}\ldots\text{I}}_{k\ \text{times}}) = \begin{cases} x^k & \text{if } k \in \mathbb{N} \setminus P \\ 2x^{k+1} & \text{if } k \in P. \end{cases} \tag{9}$$

Let us denote by $^{-}$ the inverse of ι. Thus

$$\overline{x^k} = \underbrace{\text{III}\ldots\text{I}}_{k\ \text{times}} \text{ for } k \in \mathbb{N} \setminus P, \quad \text{and} \quad \overline{2x^{k+1}} = \underbrace{\text{III}\ldots\text{I}}_{k\ \text{times}} \text{ for } k \in P.$$

Our choice of D and of the coding map $^{-}$ determines a presentation $\mathcal{P}$ of A. Specifically, $\mathcal{P}$ consists of all pairs of tuples of finite strings of strokes such that the inequality

$$\iota(\underbrace{\text{III}\ldots\text{I}}_{p_1\ \text{times}}) + \cdots + \iota(\underbrace{\text{III}\ldots\text{I}}_{p_m\ \text{times}}) \leq \iota(\underbrace{\text{III}\ldots\text{I}}_{q_1\ \text{times}}) + \cdots + \iota(\underbrace{\text{III}\ldots\text{I}}_{q_n\ \text{times}})$$

holds in G. For every $n \in \mathbb{N}$ define the pair $T_n = (v_n, w_n)$ of tuples by

$$T_n = \left((\underbrace{\text{III}\ldots\text{I}}_{n\ \text{times}}), (\underbrace{\text{III}\ldots\text{I}}_{n+1\ \text{times}}, \underbrace{\text{III}\ldots\text{I}}_{n+1\ \text{times}}) \right). \tag{10}$$

Then both (v_n, w_n) and (w_n, v_n) belong to $\mathcal{P}$ if and only if $n \in P$. From the assumed enumerability of P we get the enumerability of the subset $\mathcal{Q} = \{T_n \mid n \in P\}$. Let $\mathcal{Q}_1 \subseteq \mathcal{Q}_2 \subseteq \ldots$ be an enumerable sequence of increasing finite subsets of $\mathcal{Q}$ whose union is $\mathcal{Q}$. Since $\mathcal{M}$ is free generating in the underlying free abelian group of

$\mathcal{G}$, it follows that a pair of tuples of strings of strokes $((\overline{a_1}, \ldots, \overline{a_m}), (\overline{b_1}, \ldots, \overline{b_n}))$ belongs to $\mathcal{P}$ together with its flip $((\overline{b_1}, \ldots, \overline{b_n}), (\overline{a_1}, \ldots, \overline{a_m}))$ if and only if for some $t = 1, 2, 3, \ldots$, using the inequalities in $\mathcal{Q}_t$ and substituting equals for equals we can derive in $\mathcal{G}$ both inequalities $a_1 + \cdots + a_m \leq b_1 + \cdots + b_n$ and $b_1 + \cdots + b_n \leq a_1 + \cdots + a_m$. This shows that $\mathcal{P}$ is enumerable.

By way of contradiction, suppose $\mathcal{P}$ is decidable. Then in particular we can decide which pairs $T_n = (v_n, w_n)$ belong to $\mathcal{P}$. Thus we can decide if n is such that $v_n = w_n$, which is a contradiction with the assumed undecidability of P. This completes the proof of Theorem 2.3. □

Remark. While most AF C*-algebras with comparability existing in the literature have an enumerable presentation, uncountably many non-isomorphic AF C*-algebras with comparability do not have an enumerable presentation: this trivially follows from the existence of only countably many Turing machines.

3. Proof of Theorem 1.2

A *generalized effect algebra*, [5, 1.2.1], is a partial (associative commutative) monoid $(E, 0, +)$ satisfying (i) $a+b = a+c \Rightarrow b = c$, and (ii) $a+b = 0 \Rightarrow a = b = 0$. The stipulation $a \preceq b$ iff $\exists c \in E$ with $a+c = b$ equips E with a partial order, making 0 is the smallest element of E. We say that $(E, 0, +)$ is *totally ordered* if for any two elements $a, b \in E$ we either have $a \preceq b$ or $b \preceq a$.

This section is devoted to a proof of Theorem 1.2, stating that Elliott's totally ordered partial monoids coincide with countable totally ordered generalized effect algebras. The first step is the construction of the Baer enveloping group of E, via the following variant of [1].

Given a totally ordered generalized effect algebra $(E, 0, +)$, let us define the following binary relation $\approx$ on the set E^* of tuples of elements of E:

$$(x_1, \ldots, x_m) \approx (y_1, \ldots, y_n)$$

iff there is a matrix $(z_{ij}) := (z_{ij} : 1 \leq i \leq m,\ 1 \leq j \leq n)$ of elements of E such that $x_i = z_{i1} + \cdots + z_{in}$ and $y_j = z_{1j} + \cdots + z_{mj}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The matrix (z_{ij}) is said to be the *Riesz matrix* for $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$.

Proposition 3.1. *Let $E = (E, 0, +)$ be a totally ordered generalized effect algebra. We then have:*

(1) *E has the Riesz decomposition property, stating that, whenever $x_1 + \cdots + x_m = y_1 + \cdots + y_n$ there are $z_{ij} \in E$ such that $x_i = z_{i1} + \cdots + z_{in}$ and $y_j = z_{1j} + \cdots + z_{mj}$, for each $i = 1, \ldots, m$ and $j = 1, \ldots, n$.*

(2) *The relation $\approx$ is an equivalence relation between tuples of E.*

(3) *Denote by $\langle (x_1, \ldots, x_m) \rangle$ the equivalence class of the tuple $(x_1, \ldots, x_m)$, and set*

$$\langle (x_1, \ldots, x_m) + (y_1, \ldots, y_n) \rangle = \langle (x_1, \ldots, x_m, y_1, \ldots, y_n) \rangle.$$

Then the operation $+$ makes the set $E^*/\approx$ of equivalence classes of tuples of E into a cancellative abelian monoid

$$M = M(E) = (E^*/\approx,\ \langle 0 \rangle,\ +).$$

Writing $p \leq q$ iff there is $r \in M$ with $p+r = q$, M becomes a totally ordered monoid.

(4) *Up to isomorphism of ordered groups, M is the positive cone of a unique totally ordered abelian group $B(E)$, called the* Baer o-group of E. *Identifying E with a subset of $M = B(E)^+$ via the map, $e \mapsto \langle e \rangle$, E turns out to be a scale of $B(E)$.*

Proof. Condition (1) is easily verified, because E satisfies the condition

$$c = a + b \Rightarrow \exists a', b' \in E \text{ such that } a' \leq a, b' \leq b, c = a' + b',$$

and the Riesz decomposition property follows from this condition.

To prove (2), skipping all trivialities suppose $(x_1, \dots, x_m) \approx (y_1, \dots, y_n)$ and $(y_1, \dots, y_n) \approx (z_1, \dots, z_l)$. There are two Riesz matrices (t_{ij}) and (u_{jk}) with $i = 1, \dots, m$, $j = 1, \dots, n$, and $k = 1, \dots, l$ such that

- $x_i = t_{i1} + \cdots + t_{in}$,
- $t_{1j} + \cdots + t_{mj} = y_j = u_{j1} + \cdots + u_{jl}$
- $z_k = u_{1k} + \cdots + u_{nk}$

for all i, j, k. By (1), for each $j = 1, \dots, n$, there is a Riesz matrix $(w^j_{st}) : s = 1, \dots, m$, $t = 1, \dots, l$ such that $t_{ij} = w^j_{i1} + \cdots + w^j_{il}$ and $u_{jk} = w^j_{1k} + \cdots + w^j_{mk}$ for all $i = 1, \dots, m$ and $k = 1, \dots, l$. For fixed $i = 1, \dots, m$ and $k = 1, \dots, l$ let us now define $v_{ik} = w^1_{ik} + \cdots + w^n_{ik}$. Since $x_i = t_{i1} + t_{i2} + \cdots + t_{in} = (w^1_{i1} + \cdots + w^1_{il}) + (w^2_{i1} + \cdots + w^2_{il}) + \cdots + (w^n_{i1} + \cdots + w^n_{il})$, we see that v_{ik} exists in E. Then (v_{ik}) is a Riesz matrix for the two tuples $(x_1, \dots, x_m)$ and $(z_1, \dots, z_l)$, whence $(x_1, \dots, x_m) \approx (z_1, \dots, z_l)$. This completes the proof of (2).

To prove (3), following Baer [1], for any $w = (a_1, \dots, a_k), v = (b_1, \dots, b_l) \in E^*$, their *concatenation* is the tuple $w \smile v = (a_1, \dots, a_k, b_1, \dots, b_l)$. Let us agree to say that v is a *flip* of w if for some $i = 2, \dots, k$ the element $a = a_{i-1} + a_i$ exists in E and v is obtained replacing the two consecutive terms a_{i-1}, a_i of w by the term a. We equivalently say that w is a *flop* of v. Two tuples $v, w \in E^*$ are *equivalent*, in symbols $v \sim w$, if there is a path consisting of tuples $v_0 = v, v_1, \dots, v_{u-1}, v_u = w$ where each v_{i+1} is either a flip or a flop of v_i. Letting $[w]$ denote the equivalence class of w, the stipulation $[v] + [w] = [v \smile w]$ makes $E^*/\sim$ into a abelian monoid $N(E)$.

Claim. The two equivalence relations $\approx$ and $\sim$ are identical.

As a matter of fact, suppose $(x_1, \dots, x_m) \approx (y_1, \dots, y_n)$. There is a Riesz matrix (z_{ij}) for these two tuples, and we can write

$$(x_1, \dots, x_m) \sim (z_{11}, z_{12}, \dots, z_{1n}, z_{21}, \dots, z_{2n}, \dots z_{m1}, \dots, z_{mn}).$$

For any two tuples a, a', if a' is obtained by permuting the components of a, then $a \sim a'$, by [5, 1.7.10]. It follows that

$$(z_{11}, z_{12}, \dots, z_{1n}, z_{21}, \dots, z_{2n}, \dots, z_{m1}, \dots, z_{mn})$$

$$\sim (z_{11}, z_{21}, \dots, z_{m1}, z_{12}, \dots, z_{m2}, \dots, z_{1n}, \dots, z_{mn}) \sim (y_1, \dots, y_n),$$

and $(x_1, \dots, x_m) \sim (y_1, \dots, y_n)$. Conversely, suppose we are give two tuples

$$(x_1, \dots, x_{p-1}, x_p, x_{p+1}, \dots, x_m) \text{ and } (x_1, \dots, x_{p-1}, x_p + x_{p+1}, \dots, x_m).$$

For $i = 1, \dots, m$ and $j = 1, \dots, m-1$ we define the matrix, (z_{ij}), of elements of E by

- $z_{ij} = \delta_{ij} x_i$ if $j = 1, \dots, p-1$, or $j = p+1, \dots, m-1$ and any $i = 1, \dots, m$;

- for $j = p$, $z_{1p} = \cdots = z_{p-1,p} = z_{p+2,p} = \cdots = z_{mp} = 0$ and $z_{pp} = x_p$ and $z_{p+1,p} = x_{p+1}$.

Then (z_{ij}) is a Riesz matrix for the two tuples $(x_1, \ldots, x_{p-1}, x_p, x_{p+1}, \ldots, x_m)$ and $(x_1, \ldots, x_{p-1}, x_p + x_{p+1}, \ldots, x_m)$, whence they are $\approx$-equivalent. Assume now $(x_1, \ldots, x_m) \sim (y_1, \ldots, y_n)$. Since $\approx$ is an equivalence relation, the definition of $\sim$ is to the effect that $(x_1, \ldots, x_m) \approx (y_1, \ldots, y_n)$, and the claim is settled.

Having just proved that $M(E) = N(E)$, the proof of (3) and (4) now follows as a particular case of the general theory in [5, 1.7.6-1.7.14]. □

Proposition 3.2. *Let A be an AF C*-algebra with comparability. Let $D(A)$ be its partial Elliott monoid, $G(A)$ its dimension group, and $B(D(A))$ the Baer* o*-group of $D(A)$. We then have*

(1) *$D(A)$ is a countable totally ordered generalized effect algebra.*
(2) *Identifying $D(A)$ with a scale of B as in Proposition 3.1(4), the identity map of $D(A)$ uniquely extends to an isomorphism of the Baer o-group $B(D(A))$ onto the dimension group $G(A)$.*

Proof. (1) By [7, 5.1] or [6, chapters 7-8], $D(A)$ is a scale of $G(A)$, whence $D(A)$ is a countable partial abelian monoid satisfying the cancellativity and positivity axioms of generalized effect algebras. $D(A)$ is totally ordered because A has comparability of projections.

(2) By [6, 7.4] or [7, 4.3]. □

To conclude the proof of Theorem 1.2 we now show that every countable totally ordered generalized effect algebra E arises as the Elliott partial monoid $D(A)$ of some AF C*-algebra A with comparability.

Proposition 3.3. *Let E be a countable totally ordered generalized effect algebra, with its Baer o-group $B = B(E) \supseteq E$. Write B as the direct limit (in the category of scaled dimension groups and positive scale-preserving* one-one[4] *homomorphisms) of a countable direct system $\mathcal{S}(E)$ given by*[5]

$$\phi_n : \mathbb{Z}^{r_n}(\vec{p}(n)) \to \mathbb{Z}^{r_{n+1}}(\vec{p}(n+1)) \tag{11}$$

in such a way that $E = \bigcup \phi_{n,\infty}[0, \vec{p}_n]$.

Let $\mathcal{A}(E)$ be the norm closure of the direct limit of the system $\mathcal{S}'(E)$ of finite-dimensional C-algebras*

$$\phi_n : M(\vec{p}(n)) \to M(\vec{p}(n+1)). \tag{12}$$

We then have

(i) *For any AF C*-algebra A with comparability, $\mathcal{A}(D(A)) \cong A$.*
(ii) *Conversely, for any countable totally ordered generalized effect algebra E, $D(\mathcal{A}(E)) \cong E$.*

Proof. (i) By definition, A is the limit of a direct system

$$\psi_n : M(\vec{p}(n)) \to M(\vec{p}(n+1)) \tag{13}$$

[4]injectivity follows from the fact that B is ultrasimplicial [8, 2.2]

[5]notation from [6, p.43]

of finite-dimensional C*-algebras. Each ψ_n is a suitable rectangular matrix with integer entries ≥ 0. By Elliott's theory [7, 5.1], the direct system of scaled simplicial groups

$$\psi_n : \mathbb{Z}^{r_n}(\vec{p}(n)) \to \mathbb{Z}^{r_{n+1}}(\vec{p}(n+1)) \tag{14}$$

has the property that

$$G(A) = \lim_n \psi_n : \mathbb{Z}^{r_n} \to \mathbb{Z}^{r_{n+1}}, \tag{15}$$

and

$$D(A) = \bigcup \psi_{n,\infty}[0, \vec{p}_n]. \tag{16}$$

Since A has comparability, $G(A)$ is totally ordered and the ψ_n can be assumed to be injective, [8]. By Proposition 3.2, $D(A)$ is a totally ordered generalized effect algebra, which is a scale of $B(D(A)) \cong G(A)$. Now $\mathcal{A}(D(A))$ is defined following Elliott's construction (see the proof of [7, 5.5]) of an AF C*-algebra whose Elliott's partial monoid coincides with $D(A)$. Since both $\mathcal{A}(D(A))$ and A have isomorphic Elliott's partial monoids, by Elliott's fundamental result [7, 4.3], $\mathcal{A}(D(A)) \cong A$.

(ii) By Proposition 3.1(4) E is a scale of the countable totally ordered abelian group $B = B(E)$. The pair (B, E) satisfies the hypotheses of [7, 5.5]. Thus there is an AF C*-algebra A such that $D(A) = E$ and $G(A) = B$. By [7, 4.3], A is uniquely determined by E. Direct inspection in the proof of [7, 5.5] shows that $A = \mathcal{A}(E)$. □

We have shown that the maps $A \mapsto D(A)$ and $E \mapsto \mathcal{A}(E)$ yield a one-one correspondence between (isomorphism classes of) countable totally ordered generalized effect algebras and (isomorphism classes of) AF C*-algebras with comparability.

The proof of Theorem 1.2 is complete.

Remark. The map $A \mapsto D(A)$ is part of a covariant functor such that any isomorphism of $D(A)$ and $D(A')$ is induced by an isomorphism of A and A', [8, 5.1], but D is *not* a categorical equivalence between AF C*-algebras with comparability and countable totally ordered generalized effect algebras: e.g., every automorphism of the C*-algebra of 2×2 complex matrices is mapped by D into the identity function of $\{0, 1, 2\}$.

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