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Products of Classes of Residuated Structures

Abstract. The central result of this paper provides a simple equational basis for the join, $\mathcal{IRL} \vee \mathcal{LG}$, of the variety $\mathcal{LG}$ of lattice-ordered groups (ℓ-groups) and the variety $\mathcal{IRL}$ of integral residuated lattices. It follows from known facts in universal algebra that $\mathcal{IRL} \vee \mathcal{LG} = \mathcal{IRL} \times \mathcal{LG}$. In the process of deriving our result, we will obtain simple axiomatic bases for other products of classes of residuated structures, including the class $\mathcal{IRL} \times_s \mathcal{LG}$, consisting of all semi-direct products of members of $\mathcal{IRL}$ by members of $\mathcal{LG}$. We conclude the paper by presenting a general method for constructing such semi-direct products, including wreath products.

Keywords: residuated lattice, residuated po-monoid, direct product, semi-direct product, wreath product, lattice-ordered group, partially ordered group

1. Introduction

Let $\mathcal{IRL}$ denote the variety of integral residuated lattices, $\mathcal{IRP}$ the class of integral residuated partially ordered monoids, $\mathcal{LG}$ the variety of ℓ-groups and $\mathcal{PG}$ the class of partially ordered groups. (We refer the reader to Section 2 for the precise definitions of these concepts.) The central result of this paper provides a simple equational basis for the join $\mathcal{IRL} \vee \mathcal{LG}$. En route, we obtain simple axiomatic bases for three other classes of residuated structures,

$$\mathcal{IRP} \times \mathcal{PG},\ \mathcal{IRL} \times_s \mathcal{LG},\ \mathcal{IRP} \times_s \mathcal{PG}.$$

The preceding notation requires additional explanation. A *structure with identity* is a relational structure $\mathbf{A}$ having as a reduct a unital groupoid $(A, \cdot, e)$ such that $\{e\}$ is a subuniverse of $\mathbf{A}$. Inner direct multiplication and inner semi-direct multiplication are partial binary operations defined on the set of all substructures of a structure $\mathbf{A}$ with identity.

Definition 1.1.

1. For substructures $\mathbf{X}$ and $\mathbf{Y}$ of a structure $\mathbf{A}$ with identity, the *inner direct product* $\mathbf{X} \otimes \mathbf{Y}$ is the lattice join $\mathbf{X} \vee \mathbf{Y}$ – taken in the lattice of substructures of $\mathbf{A}$ – if the map $(x, y) \mapsto x \cdot y$ is an isomorphism from $\mathbf{X} \times \mathbf{Y}$ onto $\mathbf{X} \vee \mathbf{Y}$, but is otherwise undefined.

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2. The *direct product* $\mathcal{X} \times \mathcal{Y}$ of two classes $\mathcal{X}$ and $\mathcal{Y}$ of structures with identity of a common similarity type, is defined to be the class of all structures $\mathbf{A}$ such that $\mathbf{A} = \mathbf{X} \otimes \mathbf{Y}$ for some substructures $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$ of $\mathbf{A}$.

DEFINITION 1.2.

1. For substructures $\mathbf{X}$ and $\mathbf{Y}$ of a structure $\mathbf{A}$ with identity, the *inner semi-direct product* $\mathbf{X} \otimes_s \mathbf{Y}$ is the lattice join $\mathbf{X} \vee \mathbf{Y}$ – taken in the lattice of substructures of $\mathbf{A}$ – if there exists an endomorphism g of $\mathbf{X} \vee \mathbf{Y}$ such that $g(\mathbf{X} \vee \mathbf{Y}) = \mathbf{Y}$ and $g^{-1}(y) = Xy = yX$ for all $y \in Y$, and is undefined otherwise.
2. The *semi-direct product* $\mathcal{X} \times_s \mathcal{Y}$ of two classes $\mathcal{X}$ and $\mathcal{Y}$ of structures with identity, of a common similarity type, is defined to be the class of all structures $\mathbf{A}$ such that $\mathbf{A} = \mathbf{X} \otimes_s \mathbf{Y}$ for some substructures $\mathbf{X} \in \mathcal{X}$ and $\mathbf{Y} \in \mathcal{Y}$ of $\mathbf{A}$.

We remark that, in the familiar case of groups, $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$ if and only if $\mathbf{B}$ and $\mathbf{C}$ are normal subgroups of $\mathbf{A}$, $A = BC$ and $B \cap C = \{e\}$. Also, $\mathbf{A} = \mathbf{B} \otimes_s \mathbf{C}$ if and only if $\mathbf{B}$ is a normal subgroup of $\mathbf{A}$, $A = BC$ and $B \cap C = \{e\}$.

The primary purpose of the paper is to establish the following results.

Theorem A (See Theorem 6.2.) *An RL* $\mathbf{A}$ *belongs to* $\mathcal{IRL} \times_s \mathcal{LG}$ *if and only if it satisfies axioms (P1) and (L1).*

(P1) $e/x = x \backslash e$

(L1) $(x \backslash e \vee y \backslash e)((x \backslash e \vee y \backslash e) \backslash e) = e$

Theorem B (See Theorem 7.7.) *An RL* $\mathbf{A}$ *belongs to* $\mathcal{IRL} \times \mathcal{LG}$ *if and only if it satisfies axiom (L2).*

(L2) $(x \backslash e \vee y)((x \backslash e \vee y) \backslash e) = e$

Theorem C (See Theorem 4.5.) *An RP* $\mathbf{A}$ *belongs to* $\mathcal{IRP} \times_s \mathcal{PG}$ *if and only if it satisfies axioms (P1) and (P2).*

(P1) $e/x = x \backslash e$

(P2) $(x \backslash e)((x \backslash e) \backslash e) = e$

Theorem D (See Theorem 5.5.) *An RP* $\mathbf{A}$ *belongs to* $\mathcal{IRP} \times \mathcal{PG}$ *if and only if it satisfies axioms (P3) and (P4).*

(P3) $x(x\backslash e)(y\backslash e) = (y\backslash e)x(x\backslash e)$

(P4) $x\backslash e \leq y$ *implies* $y(y\backslash e) = e$

In Section 8, we use Theorem B above to obtain a novel proof of the fact that the variety of generalized BL algebras is a subvariety of $\mathcal{IRL} \vee \mathcal{LG}$. This result was originally obtained in [7]. The main result of Section 9, Theorem 9.4, presents necessary and sufficient conditions for constructing a semi-direct product of an IRP by a PG. We use this result in Section 10 to construct an important class of members of $\mathcal{IRP} \times_s \mathcal{PG}$, namely wreath products of IRPs by PGs.

2. Basic facts about residuated structures

We refer the reader to [3] and [14] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

A binary operation $\cdot$ on a partially ordered set $(A, \leq)$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on A such that for all $x, y, z \in A$,

$$x \cdot y \leq z \;\text{ iff }\; x \leq z/y \;\text{ iff }\; y \leq x\backslash z.$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$ and adopt the convention that, in the absence of parenthesis, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$.

The residuals may be viewed as generalized division operations, with x/y being read as "x over y" and $y\backslash x$ as "y under x". In either case, x is considered the *numerator* and y is the *denominator*. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element e. In this case, we add the monoid unit to the similarity type and refer to the resulting structure $\mathbf{A} = (A, \cdot, \backslash, /, e, \leq)$ as a *residuated partially ordered monoid* or a *residuated po-monoid* for short. If, in addition, the partial order is a lattice order, we obtain a purely algebraic structure $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, e)$ called a *residuated lattice-ordered monoid* or a *residuated lattice* for short.

Throughout this paper, the class of residuated lattices will be denoted by $\mathcal{RL}$ and that of residuated po-monoids by $\mathcal{RP}$. We adopt the convention that when a class is denoted by a string of calligraphic letters, then the members of that class will be referred to by the corresponding string of Roman letters. Thus an RL is a residuated lattice, and an RP is a residuated po-monoid.

The existence of residuals has the following basic consequences, which will be used in the remainder of the paper without explicit reference.

PROPOSITION 2.1. *Let* $\mathbf{A}$ *be an RP.*

1. *The multiplication preserves all existing joins in each argument; i.e., if* $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq A$, *then* $\bigvee_{x\in X, y\in Y}(xy)$ *exists and*

$$\Big(\bigvee X\Big)\Big(\bigvee Y\Big) = \bigvee_{x\in X, y\in Y}(xy).$$

2. *The residuals preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e. if* $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq A$, *then for any* $z \in A$, $\bigwedge_{x\in X}(x\backslash z)$ *and* $\bigwedge_{y\in Y}(z\backslash y)$ *exist and*

$$\Big(\bigvee X\Big)\Big\backslash z = \bigwedge_{x\in X}(x\backslash z) \quad\text{and}\quad z\Big\backslash\Big(\bigwedge Y\Big) = \bigwedge_{y\in Y}(z\backslash y).$$

3. *The following identities (and their mirror images) hold in* $\mathbf{A}$.

   (a) $(x\backslash y)z \le x\backslash yz$
   (b) $x\backslash y \le zx\backslash zy$
   (c) $(x\backslash y)(y\backslash z) \le x\backslash z$
   (d) $xy\backslash z = y\backslash(x\backslash z)$
   (e) $x\backslash(y/z) = (x\backslash y)/z$
   (f) $x(x\backslash x) = x$
   (g) $(x\backslash x)^2 = x\backslash x$

PROPOSITION 2.2. *A structure* $\mathbf{A} = (A, \cdot, \backslash, /, e, \le)$ *is an RP if and only if* $(A, \cdot, e, \le)$ *is a po-monoid and for all* $a, b \in A$,

(i) *the maps* $x \mapsto a\backslash x$ *and* $x \mapsto x/a$ *are isotone;*

(ii) $a(a\backslash b) \le b \le a\backslash ab$; *and*

(iii) $(b/a)a \le b \le ba/a$.

The preceding result immediately implies that the class $\mathcal{RL}$ is a finitely based variety, for, with the aid of the lattice operations, properties (i) – (iii) can be expressed as identities.

Given an RL $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, e)$ or an RP $\mathbf{A} = (A, \cdot, \backslash, /, e, \leq)$, an element $a \in A$ is said to be *integral* if $e/a = e = a\backslash e$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral. We denote by $\mathcal{IRL}$ the variety of all integral RLs, and by $\mathcal{IRP}$ the class of all integral RPs. An element $a \in A$ is said to be *invertible* if $(e/a)a = e = a(a\backslash e)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $e/a = a^{-1} = a\backslash e$. The structures in which every element is invertible are therefore precisely the ℓ-groups and the po-groups. Perhaps a word of caution is appropriate here. An *ℓ-group* is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, ^{-1}, e)$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, ^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety of $\mathcal{RL}$ defined by the equations $(e/x)x \approx e \approx x(x\backslash e)$; the term equivalence is given by $x^{-1} = e/x$ and $x/y = xy^{-1}$, $x\backslash y = x^{-1}y$. We denote by $\mathcal{LG}$ the aforementioned subvariety and refer to its members as *ℓ-groups.* Lastly, $\mathcal{PG}$ will denote the class of all po-groups.

We remark, for future reference, that $\mathcal{RL}$ is a congruence permutable variety. This can be demonstrated by the Mal'cev term $p(x, y, z) = [z \vee (z/y)x] \wedge [x \vee (x/y)z]$. Another key property – established in [3] (see also [14]) – is that each RL $\mathbf{A}$ is e-regular, that is, each congruence relation of $\mathbf{A}$ is determined by its identity block. We sketch the proof of this fact. For $a \in A$, we define the notion of *right and left conjugation by a* as follows: $\lambda_a(x) = [a\backslash(xa)] \wedge e$ and $\rho_a(x) = [(ax)/a] \wedge e$, respectively. These are unary operations on the universe of $\mathbf{A}$ that correspond to the analogous concepts from group theory. In analogy with groups, a subalgebra $\mathbf{H}$ of $\mathbf{A}$ is called *normal* if $\lambda_a(x), \rho_a(x) \in H$ for all $a \in A$ and all $x \in H$.

If Θ is a congruence relation of $\mathbf{A}$, then $[e]_\Theta$ – the Θ-block of e – is an (order) convex normal subuniverse of $\mathbf{A}$. Conversely, if H is a convex normal subuniverse of $\mathbf{A}$, then $\Theta_H = \{(a, b) : a\backslash b \wedge b\backslash a \wedge e \in H\}$ is a congruence relation of $\mathbf{A}$. Moreover, the maps $H \mapsto \Theta_H$ and $\Theta \mapsto [e]_\Theta$ are mutually inverse isomorphisms between the lattices of congruence relations and convex normal subalgebras of $\mathbf{A}$.

We close this section by calling attention to some useful applications of a basic principle from universal algebra. The principle is this: every isomorphism between two structures preserves all definable subsets, operations and relations. For example, every order isomorphism between two lattices $\mathbf{A}$ and $\mathbf{A}'$ is a lattice isomorphism:

$$h : (A, \leq) \cong (A', \leq) \text{ implies } h : \mathbf{A} \cong \mathbf{A}'.$$

PROPOSITION 2.3. *If* $\mathbf{A}$ *and* $\mathbf{A}'$ *are RLs or RPs, then*

$$h : (A, \cdot, e, \leq) \cong (A', \cdot, e, \leq) \textit{ implies } h : \mathbf{A} \cong \mathbf{A}'.$$

PROOF. The operations of residuation in an RP $\mathbf{A} = (A, \cdot, /, \backslash, e, \leq)$ are definable in the po-monoid $(A, \cdot, e, \leq)$, and the join and meet in a lattice $\mathbf{A} = (A, \vee, \wedge)$ are definable in the poset $(A, \leq)$. ■

COROLLARY 2.4. *Suppose* $\mathbf{A}$ *is an RL or an RP. For any invertible element* $c \in A$, *the map* $x \mapsto x^c = c^{-1}xc$ *is an automorphism of* $\mathbf{A}$. *(In what follows, we will refer to such a map as an inner automorphism of* $\mathbf{A}$.*)*

PROOF. We need only show that the map $\gamma_c(x) = x^c$ is an automorphism of the po-monoid $(A, \cdot, e, \leq)$. From the fact that both $\gamma_c \circ \gamma_{c^{-1}}$ and $\gamma_{c^{-1}} \circ \gamma_c$ are equal to the identity map, it follows that γ_c is bijective. Trivially, γ_c preserves the multiplication. Finally, from the fact that the multiplication is isotone it follows that γ_c is isotone, and since this is also true of its inverse $\gamma_{c^{-1}}$ we conclude that γ_c is an order isomorphism. ■

3. Facts from universal algebra

The results of this section are not used in the remainder of the paper, but they motivate and provide a context – in the realm of universal algebra – for the main results of this paper.

We start by noting that the varieties $\mathcal{LG}$ and $\mathcal{IRL}$ are disjoint, that is, their intersection is the least variety of residuated lattices. These varieties are examples of independent varieties in the sense of the following definition.

DEFINITION 3.1. ([10]) Two varieties $\mathcal{U}$ and $\mathcal{V}$, of the same similarity type, are said to be *independent* if there exists a binary term $*$ such that

$$\mathcal{U} \models v_0 * v_1 \approx v_0, \quad \mathcal{V} \models v_0 * v_1 \approx v_1.$$

It is shown in [10] that if $\mathcal{U}$ and $\mathcal{V}$ are independent varieties, then they are disjoint and $\mathcal{U} \vee \mathcal{V} = \mathcal{U} \times \mathcal{V}$. The following result provides a partial converse.

THEOREM 3.2. *If* $\mathcal{U}$ *and* $\mathcal{V}$ *are disjoint subvarieties of a congruence permutable variety, then* $\mathcal{U}$ *and* $\mathcal{V}$ *are independent and* $\mathcal{U} \vee \mathcal{V} = \mathcal{U} \times \mathcal{V}$.

PROOF. The equational theory of $\mathcal{U} \vee \mathcal{V}$ may be regarded as a congruence relation on the term algebra $\mathbf{Tm}$, say $T = Th(\mathcal{U} \vee \mathcal{V})$. By hypothesis, $\mathcal{U} \vee \mathcal{V}$ is congruence permutable, whence it follows that any two congruence relations on $\mathbf{Tm}$ that contain T permute. This applies, in particular, to the congruence relations $R = Th(\mathcal{U})$ and $S = Th(\mathcal{V})$. Thus $R \vee S = R; S$. The algebra $\mathbf{Tm}/(R \vee S)$ is in both $\mathcal{U}$ and $\mathcal{V}$, and is therefore trivial. In other words, $R \vee S = R; S$ is the universal relation. In particular, if v_0, v_1 are arbitrary variables, then $v_0 R\kappa S v_1$ for some term κ. Since the equational theories are fully invariant congruence relations, the term κ may be taken to contain no variable distinct from v_0 and v_1. Writing $v_0 * v_1$ for κ, we therefore have $v_0 R(v_0 * v_1) S v_1$, or equivalently, $\mathcal{U} \models v_0 * v_1 \approx v_0$ and $\mathcal{V} \models v_0 * v_1 \approx v_1$. ■

A procedure is described in [18] for obtaining an equational basis for the join of two independent varieties from equational bases of the joinants, assuming that the term $*$ is known. A modified version of this procedure is given below.

THEOREM 3.3. *Suppose $\mathcal{U}$ and $\mathcal{V}$ are independent varieties with equational bases U and V. Then the following equations form a basis for $\mathcal{U} \vee \mathcal{V}$.*

1. *The equations*
   (a) $(v_0 * v_1) * v_2 \approx v_0 * (v_1 * v_2)$
   (b) $(v_0 * v_1) * v_2 \approx v_0 * v_2$
   (c) $v * v \approx v$
2. *All equations of the form*
$$\lambda(v_0 * w_0, v * w_1, \cdots) \approx \lambda(v_0, v_1, \cdots) * \lambda(w_0, w_1, \cdots)$$
   with λ an operation symbol
3. *All equations $s * v \approx t * v$ with $s \approx t$ in U*
4. *All equations $v * s \approx v * t$ with $s \approx t$ in V*

PROOF. Every algebra $\mathbf{A} \in \mathcal{U} \vee \mathcal{V}$ is isomorphic to a direct product $\mathbf{B} \times \mathbf{C}$ with $\mathbf{B} \in \mathcal{U}$ and $\mathbf{C} \in \mathcal{V}$. The operation $*$ in $\mathbf{B} \times \mathbf{C}$ is given by

$$(b, c) * (b', c') = (b * b', c * c') = (b, c').$$

Under the operation $*$, $B \times C$ is therefore a rectangular band, and hence, so is A. In other words, the identities in (1) hold. Straightforward calculations show that all the identities in (2)-(4) also hold.

Suppose next that $\mathbf{A}$ is an algebra that satisfies all the identities in (1)-(4). By (1), the reduct $(A,*)$ is a rectangular band. Consequently the relations R and S defined by

$$xRy \text{ iff } x*z = y*z, \text{ for some } z \in A,$$

$$xSy \text{ iff } z*x = z*y, \text{ for some } z \in A$$

are congruence relations on $(A,*)$ and the map $h(x) = (x/R, x/S)$ is an isomorphism from $(A,*)$ onto $(A,*)/R \times (A,*)/S$. From condition (2) above we see that R and S also preserve the basic operations in $\mathbf{A}$, and hence that $h : \mathbf{A} \cong \mathbf{A}/R \times \mathbf{A}/S$. Finally, for any equation $s \approx t$ in U, $\mathbf{A} \models s*v \approx t*v$, and hence $\mathbf{A}/R \models s \approx t$, and similarly $\mathbf{A}/S \models s \approx t$ for every identity $s \approx t$ in V. ■

We observed in Section 1 that $\mathcal{RL}$ is congruence permutable, and hence so are the varieties $\mathcal{LG}$ and $\mathcal{IRL}$. It follows from Theorem 3.2 that these varieties are independent. It is actually easy to verify directly that the term $v_0 * v_1 = [v_0(v_0\backslash e)][(v_1\backslash e)\backslash e]$ satisfies the conditions of Definition 3.1. Hence, the preceding result provides a finite basis for the variety $\mathcal{IRL} \times \mathcal{LG}$. As we stated in Section 1, the primary purpose of this paper is to provide simple axiomatic bases for this variety and the three related classes of residuated structures.

4. A basis for $\mathcal{IRP} \times_s \mathcal{PG}$

For the convenience of the reader, we list here the aforementioned axioms (P1) - (P4), (L1) and (L2).

(P1) $e/x = x\backslash e$

(P2) $(x\backslash e)((x\backslash e)\backslash e) = e$

(P3) $x(x\backslash e)(y\backslash e) = (y\backslash e)x(x\backslash e)$

(P4) $x\backslash e \leq y$ implies $y(y\backslash e) = e$

(L1) $(x\backslash e \vee y\backslash e)((x\backslash e \vee y\backslash e)\backslash e) = e$

(L2) $(x\backslash e \vee y)((x\backslash e \vee y)\backslash e) = e$

The properties (P1) - (P4) do not involve the lattice operations, and can therefore be applied to RPs. Axiom (P2) states that every element of the form $a\backslash e$, for some $a \in A$, is invertible in $\mathbf{A}$. This, of course, implies that $C = \{a\backslash e : a \in A\}$ is the set of invertible elements of $\mathbf{A}$. Assuming that $\mathbf{A}$ is an RL, axiom (L1) implies (P2) and states that, in the presence of (P1), C is closed under joins. It will be shown below that properties (P4) and (L2) are equivalent, when applied to RLs, and that (L2) implies (P1) and (P3).

The main result of this section is Theorem 4.5, which asserts that the RPs in $\mathcal{IRP}\times_s\mathcal{PG}$ are precisely those satisfying (P1) and (P2). The proof of the theorem will be preceded by three lemmas. The notation below will be in effect throughout the remainder of the paper.

NOTATION 4.1. *Given an RL or an RP* $\mathbf{A}$, *we fix the following notation.*

1. $\mathbf{B}$ *is the partial substructure of* $\mathbf{A}$ *whose universe is the set of all integral elements of* $\mathbf{A}$.
2. $\mathbf{C}$ *is the partial substructure of* $\mathbf{A}$ *whose universe is the set of all invertible elements of* $\mathbf{A}$.
3. $f(a) = a(a\backslash e)$, *for all* $a \in A$.
4. $g(a) = (a\backslash e)\backslash e$, *for all* $a \in A$.

LEMMA 4.2. *Let* $\mathbf{A}$ *be an RP.*

1. *Each element* $a \in BC$ *has a unique representation* $a = bc$ *with* $b \in B$ *and* $c \in C$; *namely* $b = f(a)$ *and* $c = g(a)$.
2. $BC = CB$. *In fact, for* $b \in B$ *and* $c \in C$ *we have* $b^c \in B$ *and* $bc = cb^c$.
3. $\mathbf{B}$ *is a convex substructure of* $\mathbf{A}$ *that is invariant under all inner automorphisms by elements of* C.
4. $\mathbf{C}$ *is a substructure of* $\mathbf{A}$.
5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be substructures of* $\mathbf{A}$ *such that* $\mathbf{X}$ *is an IRP and* $\mathbf{Y}$ *is a PG. If* $A = XY$, *then* $\mathbf{X} = \mathbf{B}$ *and* $\mathbf{Y} = \mathbf{C}$.

PROOF.

(1) If $a = bc$ with b integral and c invertible, then $a\backslash e = c\backslash(b\backslash e) = c^{-1}$, hence $g(a) = c^{-1}\backslash e = c$ and $b = ac^{-1} = a(a\backslash e) = f(a)$.

(2) By Corollary 2.4, the map $x \mapsto x^c$ is an inner automorphism of $\mathbf{A}$, for all $c \in C$. This immediately implies that the set B is invariant under each such automorphism. Indeed, if $b \in B$ and $c \in C$, then $b^c\backslash e = b^c\backslash e^c = (b\backslash e)^c = e^c = e$.

(3) The set B of all integral elements of $\mathbf{A}$ is easily seen to be convex, with e as its top element. For $x, y \in B$, the element x/y lies between the

elements $x/e = x$ and $e/y = e$, and is therefore in B; by symmetry, so is $y\backslash x$.

(4) The set C of all invertible elements of $\mathbf{A}$ is obviously closed under multiplication, and the inverse c^{-1} of an invertible element c is also invertible. Closure under the residuations follows, since, for invertible elements c_0 and c_1, $c_0/c_1 = c_0c_1^{-1}$ and $c_0\backslash c_1 = c_0^{-1}c_1$.

(5) Since the IRP $\mathbf{X}$ and the PG $\mathbf{Y}$ are substructures of $\mathbf{A}$, they are substructures of $\mathbf{B}$ and $\mathbf{C}$, respectively. Noting that $A = XY$, we infer by (1) above that $\mathbf{X} = \mathbf{B}$ and $\mathbf{Y} = \mathbf{C}$. ■

LEMMA 4.3. *For any RP* $\mathbf{A}$, $\mathbf{B} \vee \mathbf{C} = \mathbf{B} \otimes_s \mathbf{C}$.

PROOF. Consider two elements $a_0 = b_0c_0$ and $a_1 = b_1c_1$, with $b_0, b_1 \in B$ and $c_0, c_1 \in C$. Note first that for all $c \in C$ and all $x, y \in A$, $c(x/y) = cx/y$ and $(y\backslash x)c = y\backslash cx$. Using these facts, easy calculations yield the following equalities:

$$\begin{aligned} a_0a_1 &= (b_o b_1^{c_0})(c_0c_1) \\ a_0\backslash a_1 &= (b_0\backslash b_1)^{c_0}\,(c_0^{-1}c_1) \\ a_1/a_0 &= (b_1/b_0^{c_0c_1^{-1}})(c_1c_0^{-1}) \end{aligned}$$

It follows that the set BC is closed under the basic operations of $\mathbf{A}$, that is, it is the universe of a substructure of $\mathbf{A}$. Obviously that substructure is $\mathbf{B} \vee \mathbf{C}$.

We next note that g is an endomorphism of $\mathbf{B} \vee \mathbf{C}$. Indeed, with a_0, a_1 as before, we have $g(a_0) = c_0$ and $g(a_1) = c_1$, while g sends a_0a_1, $a_0\backslash a_1$ and a_1/a_0 into c_0c_1, $c_0^{-1}c_1$ and $c_1c_0^{-1}$, respectively. Lastly, it is clear that $g(\mathbf{B} \vee \mathbf{C}) = \mathbf{C}$ and $g^{-1}(c) = Bc = cB$ for all $c \in C$. ■

LEMMA 4.4. *Suppose* $\mathbf{A}$ *is an RP that satisfies (P1) and (P2). Then, for all* $a \in A$,

(i) $a = f(a)g(a)$,

(ii) $f(a) \in B$ *and* $g(a) \in C$.

PROOF. By (P2), the element $g(a)$ is invertible and

$$f(a)g(a) = a(a\backslash e)((a\backslash e)\backslash e) = ae = a.$$

Also, $f(a)\backslash e = a(a\backslash e)\backslash e = (a\backslash e)\backslash(a\backslash e) = e$. The last equality holds because, by (P2), $a\backslash e$ is invertible. By (P1), $e/f(a) = e$. Hence $f(a)$ is integral. ■

We are now ready to establish the main result of the section.

THEOREM 4.5. *An RP* **A** *belongs to* $\mathcal{IRP}\times_s\mathcal{PG}$ *if and only if it satisfies (P1) and (P2).*

PROOF. If **A** belongs to $\mathcal{IRP}\times_s\mathcal{PG}$, then every element $a \in A$ is of the form $a = bc$, with b an integral element and c an invertible element. Hence e/a and $a\backslash e$ are both equal to c^{-1}, and $(a\backslash e)((a\backslash e)\backslash e) = e$. Thus (P1) and (P2) hold.

Conversely, suppose that the two identities hold. Then Lemma 4.4 implies that for all $a \in A, f(a)g(a) = a, f(a) \in B$ and $g(a) \in C$. Thus $A = BC$ and $\mathbf{A} = \mathbf{B}\otimes_s\mathbf{C}$ by Lemma 4.3. ■

5. A basis for $\mathcal{IRP} \times \mathcal{PG}$

The main result of this section, Theorem 5.5, shows that an RP is a member of $\mathcal{IRP} \times \mathcal{PG}$ if and only if it satisfies (P3) and (P4).

LEMMA 5.1. *If* **A** *is an RP in which* C *is an order filter, then for all elements* $b_0, b_1 \in B$ *and* $c_0, c_1 \in C$,

$$b_0c_0 \leq b_1c_1 \text{ iff } b_0 \leq b_1 \text{ and } c_0 \leq c_1.$$

PROOF. If $b_0c_0 \leq b_1c_1$, then $c_0c_1^{-1} \leq b_0\backslash b_1$, so $b_0\backslash b_1$ is in the filter C. The integral element $b_0\backslash b_1$ is therefore invertible, from which it follows that $b_0\backslash b_1 = e$ and $c_0c_1^{-1} \leq e$, and therefore $b_0 \leq b_1$ and $c_0 \leq c_1$. The opposite implication holds in every RP. ■

LEMMA 5.2. *An RP* **A** *belongs to* $\mathcal{IRP} \times \mathcal{PG}$ *if and only if*

1. **A** *satisfies (P1) and (P2);*
2. *the set* C *is an order filter in* **A**; *and*
3. $bc = cb$ *for all* $b \in B$ *and* $c \in C$.

PROOF. Suppose $\mathbf{A} \in \mathcal{IRP} \times \mathcal{PG}$. Then $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$. The identities (P1) and (P2) hold by Theorem 4.5, and the direct factor C is an order filter. Also, the two factors commute point-wise under the multiplication. Thus (1)-(3) hold.

Conversely, suppose (1)-(3) hold. By Theorem 4.5, $\mathbf{A} = \mathbf{B} \otimes_s \mathbf{C}$. Hence the map $h(b, c) = bc$ from $B \times C$ into A is bijective. By (3) and Lemma 5.1,

$$h : (B, \cdot, e, \leq) \times (C, \cdot, e, \leq)) \cong (A, \cdot, e, \leq),$$

whence, by Proposition 2.3, $h : \mathbf{B} \times \mathbf{C} \cong \mathbf{A}$. ■

LEMMA 5.3. *In any RP,*

(i) *If x/x is invertible, then $x/x = e$.*

(ii) *If $x/x = e = x\backslash x$, then $e/x = x\backslash e$.*

PROOF. (i) Use Proposition 2.1(3g) and the fact that invertible elements are cancellable.

(ii) We have $x(y/z) \leq xy/z$, hence, in particular, $x(e/x) \leq x/x = e$. It follows that $e/x \leq x\backslash e$. The opposite inequality follows by symmetry. ∎

Condition (P4) is obviously equivalent to the conjunction of (P2) and property (2) in Lemma 5.2. We now show that (P4) implies (P1).

COROLLARY 5.4. *Any RP that satisfies (P4) also satisfies (P1) and (P2).*

PROOF. Note first that (P2) is a special case of (P4); just let $y = x\backslash e$. Next, for $y \geq e$, (P2) and (P4) give $(y\backslash e)((y\backslash e)\backslash e) = e$ and $y(y\backslash e) = e$, respectively. It follows that $y\backslash e$ has both a left inverse and a right inverse and hence it is invertible, with inverse $y = (y\backslash e)\backslash e$. Now the elements x/x and $x\backslash x$ are always above e, and they are therefore invertible in the present situation. By the preceding lemma, this implies that $x/x = e = x\backslash x$, and hence that (P1) holds. ∎

THEOREM 5.5. *An RP* $\mathbf{A}$ *belongs to* $\mathcal{IRP} \times \mathcal{PG}$ *if and only if it satisfies (P3) and (P4).*

PROOF. (P3) and (P4) trivially hold in each of the classes $\mathcal{IRP}$ and $\mathcal{PG}$, and therefore hold in their direct product.

Assuming that $\mathbf{A}$ satisfies the two axioms, we need to show that conditions (1)-(3) in Lemma 5.2 hold. By Lemma 5.4, (P4) implies (P1) and (P2). In the presence of (P1) and (P2), conditions (2) and (3) are equivalent to (P4) and (P3), respectively. ∎

6. A basis for $\mathcal{IRL} \times_s \mathcal{LG}$

Given an RL $\mathbf{A}$, we denote by $\mathbf{A}_{rp}$ the RP reduct of $\mathbf{A}$. That is, if $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, e)$, then $\mathbf{A}_{rp} = (A, \cdot, \backslash, /, e, \leq)$. For any semi-direct decomposition $\mathbf{A} = \mathbf{X} \otimes_s \mathbf{Y}$, we obviously have $\mathbf{A}_{rp} = \mathbf{X}_{rp} \otimes_s \mathbf{Y}_{rp}$.

LEMMA 6.1. *If* $\mathbf{A}$ *is an RL, then* $\mathbf{B}$ *is a normal, convex subalgebra of* $\mathbf{A}$.

PROOF. By Lemma 4.2(3), $\mathbf{B}_{rp}$ is a convex substructure of $\mathbf{A}_{rp}$. Thus, all we need to show is that the set B is closed under the lattice operations and conjugation.

For any $x, y \in B$, we have $x, y \leq e$, and hence $xy \leq x \wedge y \leq x \vee y \leq e$. Since xy and e are in B, it follows that $x \wedge y$ and $x \vee y$ are in B. To prove closure under conjugation, it suffices by symmetry to show that, for all $x \in B$ and $a \in A$, $a \backslash xa \in B$, since B consists of elements below e. Representing a as a product, $a = bc$, with $b \in B$ and $c \in C$, we have $a \backslash xa = c^{-1}(b \backslash xb)c$. The element $b \backslash xb$ is in B because both x and b are in B, and any inner automorphism (by the invertible element c) is an automorphism of $\mathbf{A}$, and therefore sends the definable subset B into itself. ∎

THEOREM 6.2. *An RL* $\mathbf{A}$ *belongs to* $\mathcal{IRL} \times_s \mathcal{LG}$ *if and only if* $\mathbf{A}$ *satisfies (P1) and (L1).*

PROOF. Suppose

$$\mathbf{A} \in \mathcal{IRL} \times_s \mathcal{LG}. \tag{6.1}$$

Then $\mathbf{A} = \mathbf{B} \otimes_s \mathbf{C}$. Hence the sets B and C are closed under all the basic operations in $\mathbf{A}$. In particular,

$$C \text{ is closed under joins.} \tag{6.2}$$

Also, $\mathbf{A}_{rp} \in \mathcal{IRP} \times_s \mathcal{PG}$, so (P1) and (P2) hold in $\mathbf{A}_{rp}$, and therefore also in $\mathbf{A}$. Since (L1) is equivalent to the conjunction of (P2) and (6.2), we conclude that (P1) and (L1) hold in $\mathbf{A}$.

Now assume that $\mathbf{A}$ satisfies (P1) and (L1). To prove that (6.1) holds, it suffices to show the following:

$$B \text{ and } C \text{ are subuniverses of } \mathbf{A}. \tag{6.3}$$

$$g \text{ is an endomorphism of } \mathbf{A}. \tag{6.4}$$

$$g(A) = C. \tag{6.5}$$

$$g^{-1}(c) = Bc, \text{ for all } c \in C. \tag{6.6}$$

By Lemma 6.1, it is true for any RL $\mathbf{A}$ that B is a subuniverse of $\mathbf{A}$ and by Lemma 4.2(4), C is always closed under the multiplication and the residuals. In an RL that satisfies (P1) and (P2), a direct check shows that g is a closure operation with C as its set of closed elements, and C is therefore closed under meets. Finally, the closure of C under joins is guaranteed by (L1). Thus (6.3) holds.

By Lemma 6.1, $\mathbf{B}$ is a normal, convex subalgebra of $\mathbf{A}$. The elements of the quotient algebra $\mathbf{A}$ are the blocks Ba with $a \in A$. From the fact that $A = BC$ it follows that each block is equal to Bc for some $c \in C$, and from the fact that each element $a \in A$ has a unique representation $a = bc$ with $b \in B$ and $c \in C$ we see that each block contains a unique member of C. Using the fact that $\mathbf{C}$ is a subalgebra of $\mathbf{A}$, we conclude that (6.4), (6.5) and (6.6) hold. ■

7. A basis for $\mathcal{IRL} \times \mathcal{LG}$

Our main result in this section states that adding (L2) to an equational basis for $\mathcal{RL}$ yields an equational basis for the product (join) of $\mathcal{IRL}$ and $\mathcal{LG}$. This result generalizes an earlier result in [6], which presents a finite, but more involved, equational basis for this variety.

LEMMA 7.1. *An RL* $\mathbf{A}$ *belongs to* $\mathcal{IRL} \times \mathcal{LG}$ *if and only if* $\mathbf{A}$ *satisfies (L2) and (P3).*

PROOF. The two identities hold in the varieties $\mathcal{IRL}$ and $\mathcal{LG}$, and hence also in their direct product.

Now consider an RL that satisfies (L2) and (P3). We are going to apply Proposition 2.3 with $\mathbf{A}' = \mathbf{B} \times \mathbf{C}$. In order to make sure that this theorem applies, we note that, by Theorems 6.2 and 5.5,

$$\mathbf{A} = \mathbf{B} \otimes_s \mathbf{C}; \text{ and} \tag{7.1}$$

$$\mathbf{A}_{rp} = \mathbf{B}_{rp} \otimes \mathbf{C}_{rp}. \tag{7.2}$$

This is true because (L2) implies (P1), (L1) and (P4). Implicit in (7.1) is the fact that the partial subalgebras $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}$ are actually subalgebras. By (7.2), the map $h(b, c) = bc$ is an isomorphism from $\mathbf{B}_{rp} \times \mathbf{C}_{rp}$ onto $\mathbf{A}_{rp}$, and we infer, by Proposition 2.3, that $h : \mathbf{B} \times \mathbf{C} \cong \mathbf{A}$ i. e., that $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$. ■

Our next goal is to show that (P3) can be omitted. In other words, we are going to prove that if (L2) holds, then every member b of B commutes with every member c of C.

LEMMA 7.2. *Suppose* $\mathbf{A}$ *is an RL that satisfies (L2). For any* $b_0, b_1 \in B$ *and* $c_0, c_1 \in C$,

$$\begin{aligned} b_0c_0 \vee b_1c_1 &= (b_0 \vee b_1)(c_0 \vee c_1), \\ b_0c_0 \wedge b_1c_1 &= (b_0 \wedge b_1)(c_0 \wedge c_1). \end{aligned}$$

PROOF. We have previously remarked that (L2) implies (P1) and (L1). By Theorem 6.2, $A = BC$. Let $a = bc$ be an arbitrary element of A with $b \in B$ and $c \in C$. By Lemma 5.1, we have for $i = 0, 1$, $b_i c_i \le bc$ iff $b_i \le b$ and $c_i \le c$. Hence, $b_0 c_0 \vee b_1 c_1 \le bc$ iff $b_0 \vee b_1 \le b$ and $c_0 \vee c_1 \le c$. A similar argument applies for meets. ■

COROLLARY 7.3. *Suppose* **A** *is an RL that satisfies (L2). For any* $b \in B$ *and* $c \in C$, $b \wedge c = b(c \wedge e) = (c \wedge e)b$.

PROOF. We have, by Lemma 7.2, $b(c \wedge e) = (b \wedge e)(e \wedge c) = be \wedge ec = b \wedge c$. This proves the first equality, the second one follows by symmetry. ■

LEMMA 7.4. *Suppose* **A** *is an RL that satisfies (L2). For any* $b \in B$ *and* $c \in C$, *if* c *is comparable with* e, *then* $bc = cb$.

PROOF. For $c \le e$, both bc and cb are equal to $b \wedge c$ by the preceding corollary. If $c \ge e$, then $c^{-1} \le e$. In this case, c^{-1} commutes with b, and hence so does c. ■

LEMMA 7.5. *Suppose* **A** *is an RL that satisfies (L2). For all* $a \in A$,

$$a = (a \vee e)(a \wedge e).$$

PROOF. Using the fact that the element $a \vee e$ is invertible, write the equation in the equivalent form

$$(a \vee e)^{-1} a = a \wedge e.$$

Invoking Corollary 7.3 and writing a in the usual form, $a = bc$, with $b \in B$ and $c \in C$, we compute:

$$\begin{aligned} (a \vee e)^{-1} a &= ((bc \vee e)\backslash e)bc = (bc\backslash e \wedge e)\, bc \\ &= (c^{-1} \wedge e)bc = b(c^{-1} \wedge e)c \\ &= b(e \wedge c) = (b \wedge e)(c \wedge e) \\ &= bc \wedge ee = a \wedge e. \end{aligned}$$

■

LEMMA 7.6. *Every RL that satisfies (L2) also satisfies (P3).*

PROOF. Suppose **A** is an RL that satisfies (L2), and consider any $c \in C$. Then $c = (c \vee e)(c \wedge e)$. The elements $c \vee e$ and $c \wedge e$ are both invertible, since C is a subuniverse by Theorem 6.2, and they are both comparable with e. Hence, by Lemma 7.4, they both commute with every member of B. It follows that their product c also commutes with every member of B. ■

THEOREM 7.7. *An RL* $\mathbf{A}$ *belongs to* $\mathcal{IRP} \times \mathcal{LG}$ *if and only if it satisfies (L2).*

PROOF. By Lemmas 7.1 and 7.6. ∎

COROLLARY 7.8. *Suppose* $\mathcal{U}$ *is a variety of RLs that satisfies (L2). Then*

$$\mathcal{U} = (\mathcal{U} \cap \mathcal{IRL}) \times (\mathcal{U} \cap \mathcal{LG}).$$

PROOF. We have $\mathcal{U} \subseteq \mathcal{IRL} \times \mathcal{LG}$. For any $\mathbf{A} \in \mathcal{U}$, the subalgebras $\mathbf{B}$ and $\mathbf{C}$ of $\mathbf{A}$ are in $\mathcal{IRL}$ and $\mathcal{LG}$, respectively. From the fact that $\mathbf{A} = \mathbf{B} \otimes \mathbf{C}$ it follows that $\mathbf{A} \in \mathcal{U}$ if and only if $\mathbf{B}, \mathbf{C} \in \mathcal{U}$. Therefore,

$$\mathbf{A} \in \mathcal{U} \text{ iff } \mathbf{B} \in \mathcal{U} \cap \mathcal{IRL} \text{ and } \mathbf{C} \in \mathcal{U} \cap \mathcal{LG}.$$

∎

8. An application

The notion of a generalized MV-algebra, introduced in [2] and [14], generalizes the classical notion of an MV-algebra in the context of residuated lattices to include non-commutative, non-integral and unbounded structures. The class of generalized MV-algebras includes ℓ-groups, their negative cones, generalized Boolean algebras and classical MV-algebras. It is shown in [7] that generalized MV-algebras can be obtained from ℓ-groups via a truncation construction that subsumes the Chang-Mundici Γ functor. Moreover, this correspondence extends to a categorical equivalence that generalizes the ones established in [16] (see also [4]) and [5].

A significant ingredient of the aforementioned result is the observation that the variety of generalized MV-algebras – in fact, the more general class of generalized BL-algebras – is a subvariety of $\mathcal{IRL} \vee \mathcal{LG}$. The aim of this section is to provide a proof of this result based on the theory developed in the preceding sections.

DEFINITION 8.1. By a *generalized basic logic algebra*, or a GBLA for short, we mean an RL that satisfies the identities

$$x(x\backslash(x \wedge y)) = x \wedge y = ((x \wedge y)/x)x. \tag{8.1}$$

We denote by $\mathcal{GBLA}$ the variety of all GBLAs.

We remark that the identities (8.1) are obviously equivalent to the quasi-identities

$$x \geq y \text{ implies } x(x\backslash y) = y = (y/x)x. \tag{8.2}$$

Another equivalent identity, which will not be needed in the sequel, is

$$x(x\backslash y \wedge e) = x \wedge y = (y/x \wedge e)x. \tag{8.3}$$

(see [7] for details).

The quasi-identities (8.2) imply (just let $y = e$):

Every positive element of a GBLA is invertible. (8.4)

LEMMA 8.2. ([7]) *Every GBLA satisfies the identity*

$$x = (x \vee e)(x \wedge e). \tag{8.5}$$

PROOF. Using the fact that $x \vee e$ is invertible, we write the identity in the equivalent form

$$(x \vee e)^{-1}x = x \wedge e.$$

We now compute,

$$(x \vee e)^{-1}x = (e/(x \vee e))x = (e/x \wedge e)x = x \wedge e.$$

■

We are ready to prove the main result of this section.

THEOREM 8.3. ([7]) *Let $\mathcal{U}$ be the variety of all IRLs that satisfy the identities*

$$x(x\backslash y) = x \wedge y = (y/x)x. \tag{8.6}$$

Then $\mathcal{GBLA} = \mathcal{U} \times \mathcal{LG}$.

PROOF. It will suffice to prove that every GBLA satisfies (L2). Indeed, this will imply, by Corollary 7.8, that

$$\mathcal{GBLA} = (\mathcal{GBLA} \cap \mathcal{IRL}) \times (\mathcal{GBRL} \cap \mathcal{LG}).$$

The proof is completed by noting that every LG is a GBLA, while an IRL is a GBLA if and only if it satisfies the identity (8.6).

Consider a GBLA **A**. We proceed to show that **A** satisfies (L2). We have noted that every positive element of A is invertible. We first show that, for every $x \in A$, the elements $x\backslash e$ and e/x are invertible. By symmetry, we only need consider $x\backslash e$. We have, by (8.5),

$$x\backslash e = (x\backslash e \vee e)(x\backslash e \wedge e).$$

The first factor is positive, and therefore invertible. The second factor, $x\backslash e \wedge e = (x \vee e)\backslash e$, is the inverse of the invertible element $x \vee e$, and is therefore also invertible. Hence $x\backslash e$ is invertible. Lastly, consider the join $z = x\backslash e \vee y$, for $x, y \in A$. By (8.2), $z(z\backslash(x\backslash e)) = x\backslash e$. Thus, $z(xz\backslash e) = x\backslash e$, showing that z is invertible, since both $xz\backslash e$ and $x\backslash e$ are invertible. This establishes (L2) and completes the proof of the theorem. ∎

DEFINITION 8.4. By a *generalized MV-algebra*, or a GMVA for short, we mean an RL that satisfies the identities

$$x/((x \vee y)\backslash x) = x \vee y = (x/(x \vee y))\backslash x. \tag{8.7}$$

We denote by $\mathcal{GMVA}$ the variety of all GMVAs.

The identities 8.7 are equivalent to the following quasi-identities:

$$x \leq y \text{ implies } x/(y\backslash x) = y = (x/y)\backslash x. \tag{8.8}$$

The identities (8.7) easily imply that an IRL is a GMVA if and only if it satisfies the identities

$$x/(y\backslash x) = x \vee y = (x/y)\backslash x. \tag{8.9}$$

It is shown in [7] that $\mathcal{GMVA}$ is a subvariety of $\mathcal{GBLA}$. Thus, the preceding result implies the final result of this section.

COROLLARY 8.5. ([7]) *Let $\mathcal{U}$ be the variety of all IRLs that satisfy the identities (8.9). Then $\mathcal{GMVA} = \mathcal{U} \times \mathcal{LG}$.*

9. Constructing semi-direct products

We established in Theorem 4.5 that an RP belongs to $\mathcal{IRP} \times_s \mathcal{PG}$ if and only if it satisfies axioms (P1) and (P2). Unlike a direct product, a semi-direct product is not uniquely determined by its factors, as it depends on how these factors interact with each other. Thus, it is important to have a general method for constructing such products. The main result of this section, Theorem 9.4, presents necessary and sufficient conditions for constructing a semi-direct product of an IRP by a PG.

We start with a review of the simple case of monoids. Let $\mathbf{A}$ be a semi-direct product of a monoid $\mathbf{B}$ by a group $\mathbf{C}$. Then, by Definition 1.2, there exists a homomorphism $g : \mathbf{A} \to \mathbf{C}$ with $g^{-1}(c) = Bc = cB$, for all $c \in C$. In light of Lemma 4.2, every element $a \in A$ has a unique "standard representation" as a product $a = bc$, with $b \in B$ and $c \in C$. The same result also tells us that the submonoid $\mathbf{B}$ is closed under inner automorphisms by elements of $\mathbf{C}$. Hence the only information we need to reconstruct the multiplication table of $\mathbf{A}$ from the two factors is the action of $\mathbf{C}$ on $\mathbf{B}$, as reflected by such automorphisms. Indeed, observe that if $a_0 = b_0c_0$ and $a_1 = b_1c_1$ are elements in standard form, then the standard form for the product a_0a_1 is $a_0a_1 = b_0{b_1}^{c_0^{-1}}c_0c_1$, with $b_0{b_1}^{c_0^{-1}} \in B$ and $c_0c_1 \in C$.

The following result is almost immediate.

LEMMA 9.1. *Given a monoid* $\mathbf{B}$ *and a group* $\mathbf{C}$ *with* $B \cap C = \{e\}$, *together with a homomorphism* $\gamma : \mathbf{C} \to Aut(\mathbf{B})$, *there exists, up to isomorphism, a unique semi-direct product* $\mathbf{A}$ *of* $\mathbf{B}$ *by* $\mathbf{C}$ *such that* $b^{c^{-1}} = \gamma_c(b)$, *for all* $c \in C$ *and* $b \in B$. *(Throughout the paper,* γ_c *will denote the value of* γ *at* $c \in C$.*)*

DEFINITION 9.2. If $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and γ are as in the preceding lemma, then we say that $\mathbf{A}$ is the semi-direct product of $\mathbf{B}$ by $\mathbf{C}$ *induced by* γ, and we write $\mathbf{A} = \mathbf{B} \times_\gamma \mathbf{C}$.

Given an IRP $\mathbf{B}$ and a PG $\mathbf{C}$, we can apply the preceding lemma to the monoid reducts $\mathbf{B}_m = (B, \cdot, e)$ and $\mathbf{C}_m = (C, \cdot, e)$. For each homomorphism $\gamma : \mathbf{C}_m \to Aut(\mathbf{B}_m)$, this yields a monoid $\mathbf{A}_m = \mathbf{B}_m \times_\gamma \mathbf{C}_m$. However, in general $\mathbf{A}_m$ cannot be expanded to a partially ordered monoid, because the monoid automorphisms γ_c need not be isotone. We therefore restrict the choice of these automorphisms by requiring that $\gamma : \mathbf{C}_m \to Aut(\mathbf{B})$ be a homomorphism. That is, we stipulate that each γ_c be an automorphism of the RP $\mathbf{B}$, not just of the monoid $\mathbf{B}_m$.

We wish to construct partial orders $\leq$ of A that agree with the given partial orders on B and C, and with respect to which multiplication on A is residuated. We further require that the residuals on A agree with those on B and C. Thus for $b_0,\ b_1 \in B$ and $c_0,\ c_1 \in C$, we stipulate that

$$b_0c_0 \leq b_1c_1 \text{ iff } c_0c_1^{-1} \leq b_0\backslash b_1.$$

The preceding equivalence shows that the partial orders can be completely defined by specifying, for each $c \in C$, the set $F(c)$ of elements of B that are above c. To facilitate the discussion, we define a relation $\leq_F$ for an arbitrary map $F : C \to \mathcal{P}(B)$, and then list the conditions that F must satisfy in order for $\leq_F$ to have the required properties.

DEFINITION 9.3. Suppose $\mathbf{B}$ is an IRP and $\mathbf{C}$ is a PG. Let $\gamma : \mathbf{C}_m \to Aut(\mathbf{B})$ be a homomorphism and let $\mathbf{A}_m = \mathbf{B}_m \times_\gamma \mathbf{C}_m$. For any map $F : C \to \mathcal{P}(B)$, and for all $b_0, b_1 \in B$, $c_0, c_1 \in C$, we define

$$b_0c_0 \leq_F b_1c_1 \text{ iff } b_0\backslash b_1 \in F(c_0c_1^{-1}). \tag{9.1}$$

If $\leq_F$ is a partial order and the multiplication in $\mathbf{A}_m$ is residuated relative to $\leq_F$, then we say that F is *admissible* for $\mathbf{B}$, $\mathbf{C}$ and γ, and denote by $\mathbf{B} \times_{\gamma,F} \mathbf{C}$ the induced RP.

Every semi-direct product of an IRP $\mathbf{B}$ by a PG $\mathbf{C}$ is obtained in this manner. The following result characterizes all admissible maps F.

THEOREM 9.4. *Suppose* $\mathbf{B}$ *is an IRP,* $\mathbf{C}$ *a PG and* $\gamma : \mathbf{C}_m \to Aut(\mathbf{B})$ *a homomorphism. Let* $\mathbf{A}_m = \mathbf{B}_m \times_\gamma \mathbf{C}_m$. *A map* $F : C \to \mathcal{P}(B)$ *is admissible if and only if the following conditions hold for all* $b, b_0, b_1 \in B$ *and* $c, c_0, c_1 \in C$.

(i) *If* $c \not\leq e$, *then* $F(c) = \emptyset$.

(ii) *If* $c \leq e$ *then* $F(c)$ *is an order filter in* $\mathbf{B}$.

(iii) $F(e) = \{e\}$.

(iv) *If* $c_0 \leq c_1$, *then* $F(c_0) \supseteq F(c_1)$.

(v) $F(c_0)F(c_1) \subseteq F(c_0c_1)$.

(vi) $F(c_0^{c_1}) = F(c_0)^{c_1}$.

(vii) $b_0\backslash b_1 \in F(c)$ *iff* $b_1/b_0^c \in F(c)$.

PROOF. Suppose F is admissible. Then $\leq_F$ is a partial order that agrees with the partial orders on $\mathbf{B}$ and $\mathbf{C}$. For $c \in C$, $F(c)$ is the set of all elements of B that are above c, and using the fact that e is the top element of $\mathbf{B}$, we infer that (i)-(iv) hold. Also, the multiplication must be isotone relative to $\leq$, from which it follows that (v) holds. The proofs of (vi) and (vii) are straightforward. For example, the following calculation establishes (vii):

$$b_0\backslash b_1 \in F(c) \text{ iff } c \leq b_0\backslash b_1 \text{ iff } b_0c \leq b_1 \text{ iff } cb_0{}^c \leq b_1 \text{ iff } c \leq b_1/b_0^c.$$

Conversely, suppose (i)-(vii) hold. For $b_0, b_1 \in B$ we have

$$b_0 \leq_F b_1 \text{ iff } b_0e \leq_F b_1e \text{ iff } b_0\backslash b_1 \in F(e) \text{ iff } b_0\backslash b_1 = e \text{ iff } b_0 \leq b_1,$$

while for $c_0, c_1 \in C$,

$$c_0 \leq_F c_1 \text{ iff } ec_0 \leq_F ec_1 \text{ iff } e\backslash e \in F(c_0c_1^{-1}) \text{ iff } c_0c_1^{-1} \leq e \text{ iff } c_0 \leq c_1.$$

The relation $\leq_F$ therefore agrees with the partial orderings of **B** and **C**. We henceforth drop the subscript F, writing $\leq$ for $\leq_F$.

We next prove that $\leq$ is a partial order.

Reflexivity: Note that $b_0c_0 \leq b_0c_0$ if and only if $b_0\backslash b_0 \in F(c_0c_0^{-1}) = F(e) = \{e\}$, that is, if and only if $b_0\backslash b_0 = e$. Since the latter equality is always true for integral elements, we have that $\leq$ is reflexive.

Antisymmetry: If $b_0c_0 \leq b_1c_1 \leq b_0c_0$, then $b_0\backslash b_1 \in F(c_0c_1^{-1})$ and $b_1\backslash b_0 \in F(c_1c_0^{-1})$, hence $(b_0\backslash b_1)(b_1\backslash b_0) \in F(c_0c_1^{-1})F(c_1c_0^{-1}) \subseteq F(e) = \{e\}$. Thus $(b_0\backslash b_1)(b_1\backslash b_0) = e$, which implies that $b_0\backslash b_1 = e = b_1\backslash b_0$, hence $b_0 = b_1$. Therefore, $\leq$ is antisymmetric.

Transitivity: If $b_0c_0 \leq b_1c_1 \leq b_2c_2$, then $b_0\backslash b_1 \in F(c_0c_1^{-1})$ and $b_1\backslash b_2 \in F(c_1c_2^{-1})$ so that $(b_0\backslash b_1)(b_1\backslash b_2) \in F(c_0c_1^{-1})F(c_1c_2^{-1}) \subseteq F(c_0c_2^{-1})$. Using the fact that $(b_0\backslash b_1)(b_1\backslash b_2) \leq b_0\backslash b_2$, we infer that $b_0\backslash b_2 \in F(c_0c_2^{-1})$, and hence $b_0c_0 \leq b_2c_2$. Therefore, $\leq$ is transitive.

We next prove that **A** is a po-monoid. To show that an inequality $x \leq y$ is preserved when both sides are multiplied by the same element z, either on the left or on the right, it suffices to consider the cases when the multiplier is either a member of B or a member of C. We establish isotonicity for multiplication by elements of B and leave to the reader the simpler cases when the multiplier is a member of C. In the case of left multiplication by an element $z = b \in B$, we represent x and y in the standard form, $x = b_0c_0$ and $y = b_1c_1$. Then $zx = (bb_0)c_0$ and $zy = (bb_1)c_1$. Given that $b_0\backslash b_1 \in F(c_0c_1^{-1})$, we therefore need to show that $(bb_0)\backslash(bb_1) \in F(c_0c_1^{-1})$. But this follows from the inequality $b_0\backslash b_1 \leq (bb_0)\backslash(bb_1)$, which holds in every RP. For right multiplication by an element w of B, we proceed similarly, using the dual representations $x = c_0b_0$ and $y = c_1b_1$, and using in place of (9.1) the equivalence

$$c_0b_0 \leq c_1b_1 \text{ iff } b_1/b_0 \in F(c_1^{-1}c_0). \tag{9.2}$$

Thus we can complete the proof of this case by verifying (9.2). To this end, first observe that for all $b_0, b_1 \in B$ and all $c \in C$,

$$(b_1/b_0)^c = b_1^c/b_0^c. \tag{9.3}$$

This simply reflects the facts that $\gamma_c(b) = b^c$, for all $b \in B$ and all $c \in C$, and that each γ_c is an automorphism of the RP **B**. We use (9.3) to verify (9.2):

$$c_0b_0 \leq c_1b_1 \text{ iff } b_0^{c_0^{-1}}c_0 \leq b_1^{c_1^{-1}}c_1 \text{ iff } b_0^{c_0^{-1}}\backslash b_1^{c_1^{-1}} \in F(c_0c_1^{-1}) \ \ \{\text{by } (9.1)\} \text{ iff}$$

$b_1^{c_1^{-1}}/(b_0^{c_0^{-1}})^{c_0c_1^{-1}} \in F(c_0c_1^{-1})$ {by (vii)} iff $b_1^{c_1^{-1}}/b_0^{c_1^{-1}} \in F(c_0c_1^{-1})$ iff $(b_1^{c_1^{-1}}/b_0^{c_1^{-1}})^{c_1} \in F(c_0c_1^{-1})^{c_1}$ iff $b_1/b_0 \in F(c_1^{-1}c_0)$ {by (9.3) and (vi)}.

Lastly, we need to establish that A is an RP. Given $a_0, a_1 \in A$, we show that the inequality $a_0x \leq a_1$ has a largest solution x. With a_0, a_1 and x in standard form – $a_0 = b_0c_0$, $a_1 = b_1c_1$ and $x = yz$ – the inequality is equivalent to $b_0c_0yz \leq b_1c_1$, which in turn is equivalent to $b_0y^{c_0^{-1}}c_0z \leq b_1c_1$. Hence $a_0x \leq a_1$ holds if and only if

$$b_0y^{c_0^{-1}}\backslash b_1 \in F(c_0zc_1^{-1}). \tag{9.4}$$

For any solution $x = yz$ of (9.4), we have $c_0zc_1^{-1} \leq e$, hence $z \leq c_0^{-1}c_1$. We get one solution $x_0 = y_0z_0$ by taking $z_0 = c_0^{-1}c_1$ and $y_0 = (b_0\backslash b_1)^{c_0}$. This is in fact easily seen to be a maximal solution, so our problem now reduces to showing that, every other solution satisfies the inequality $x \leq x_0$. Assuming 9.4, we therefore need to show that $yz \leq (b_0\backslash b_1)^{c_0}c_0^{-1}c_1$ or equivalently,

$$y\backslash(b_0\backslash b_1)^{c_0} \in F(zc_1^{-1}c_0). \tag{9.5}$$

To see that (9.4) implies (9.5), we observe that

$$y\backslash(b_0\backslash b_1)^{c_0} = (b_0y^{c_0^{-1}}\backslash b_1)^{c_0}, \quad \text{by (9.3)},$$

$$zc_1^{-1}c_0 = (c_0zc_1^{-1})^{c_0},$$

and invoke (vi). Thus the left residuals exist. The existence of right residuals follows similarly from (9.2) and the mirror equation of (9.3). ∎

We will use the preceding result in the construction of the wreath product in the next section.

10. Wreath products

The purpose of this section is to introduce the important concept of a wreath product, a special type of a semi-direct product of an IRP by a PG. As an application, we will use this construction to exhibit examples of RLs in $\mathcal{IRL} \times_s \mathcal{LG}$ that are not members of $\mathcal{IRL} \times \mathcal{LG}$.

The concept of a wreath product of two ℓ-groups, with the second factor a totally ordered group, was introduced in [17]. Our concept of a wreath product is a variant of the more general concept of a wreath product of two ℓ-groups first considered in [15] and generalized for an arbitrary family of ℓ-groups in [13]. A detailed account of this important construction is presented in [8] and [9].

The second factor, $\mathbf{C}$, of a wreath product $\mathbf{D}$ *Wr* $\mathbf{C}$ is a PG of automorphisms of a poset $\mathbf{X} = (X, \leq)$. We remark that multiplication in $\mathbf{C}$ is composition of functions and the partial order is defined point-wise: $\sigma \leq \tau$ if and only if $\sigma(x) \leq \tau(x)$, for all $x \in X$. The first factor $\mathbf{B}$ is taken to be the X-th power $\mathbf{D}^X$ of an IRP $\mathbf{D}$. The universe of the wreath product $\mathbf{D}$ *Wr* $\mathbf{C}$ is the cartesian product $B \times C$. Its multiplicative reduct is the semidirect product $\mathbf{B} \otimes_s \mathbf{C}$ induced by the homomorphism $\gamma : \mathbf{C} \to Aut(\mathbf{B})$ defined by $\gamma_\sigma(f) = f \circ \sigma^{-1}$, for all $\sigma \in C$ and $f \in D^X$. Let $\hat{e} \in B$ denote the constant map with image $\{e\}$, and let $\mathbf{1}$ denote the identity map on X. In what follows, we will identify $\mathbf{B}$ and $\mathbf{C}$ with their isomorphic copies $\mathbf{B}' = \mathbf{B} \times \{\mathbf{1}\}$ and $\mathbf{C}' = \{\hat{e}\} \times \mathbf{C}$, respectively.

The partial ordering on the wreath product will be defined in terms of a map F introduced in the previous section. For $(\hat{e}, \sigma) \in C'$, we let $F(\hat{e}, \sigma) = \emptyset$ if $(\hat{e}, \sigma) \not\leq (\hat{e}, \mathbf{1})$, that is, if $\sigma \not\leq \mathbf{1}$. If $(\hat{e}, \sigma) \leq (\hat{e}, \mathbf{1})$, we let $F(\hat{e}, \sigma) = \{(f, \mathbf{1}) : \forall x \in X,\ \sigma(x) = x \Rightarrow f(x) = e\}$.

THEOREM 10.1. *If* $\mathbf{D}$ *is an IRP and* $\mathbf{C}$ *is a PG, then* $\mathbf{D}$ *Wr* $\mathbf{C}$ *is an RP.*

PROOF. We will prove that the map F is admissible for $\mathbf{B}'$, $\mathbf{C}'$ and γ, by verifying conditions (i)-(vii) in Theorem 9.4.

(i) By definition, $F(\hat{e}, \sigma) = \emptyset$ if $(\hat{e}, \sigma) \not\leq (\hat{e}, \mathbf{1})$.

(ii) Suppose $(f, \mathbf{1}) \in F(\hat{e}, \sigma)$ and $(f, \mathbf{1}) \leq (g, \mathbf{1})$. For every fixed point x of σ we then have $f(x) = e$, and hence $g(x) = e$, so $(g, \mathbf{1}) \in F(\hat{e}, \sigma)$.

(iii) Every member of X is a fixed point of $\mathbf{1}$, whence $(\hat{e}, \mathbf{1})$ is the only member of $F(\hat{e}, \mathbf{1})$.

(iv) Let $(\hat{e}, \sigma) \leq (\hat{e}, \tau)$. We may assume that $(\hat{e}, \tau) \leq (\hat{e}, \mathbf{1})$. Then $\sigma(x) \leq \tau(x) \leq x$ for every $x \in X$. From this it follows that every fixed point of σ is also a fixed point of τ, whence the inclusion $F(\hat{e}, \sigma) \supseteq F(\hat{e}, \tau)$ follows.

(v) Assuming that $(f, \mathbf{1}) \in F(\hat{e}, \sigma)F(\hat{e}, \tau)$, we have that $(f, \mathbf{1}) = (g, \mathbf{1})(h, \mathbf{1}) = (gh, \mathbf{1})$, with $(g, \mathbf{1}) \in F(\hat{e}, \sigma)$ and $(h, \mathbf{1}) \in F(\hat{e}, \tau)$. We need to show that $(f, \mathbf{1}) \in F((\hat{e}, \sigma)(\hat{e}, \tau)) = F(\hat{e}, \sigma \circ \tau)$. We claim that every fixed point x of $\sigma \circ \tau$ is a fixed point of both σ and τ. Indeed, since the sets $F(\hat{e}, \sigma)$ and $F(\hat{e}, \tau)$ are non-empty, σ and τ are both below $\mathbf{1}$. Hence, $x = \sigma \circ \tau(x) = \sigma(\tau(x)) \leq \tau(x) \leq x$, from which it follows that $\tau(x) = x$ and $\sigma(x) = \sigma(\tau(x)) = x$. From this claim it follows that if x is a fixed point of $\sigma \circ \tau$, then $g(x) = e$ and $h(x) = e$, and hence $f(x) = g(x)h(x) = e$.

(vi) Consider elements $(f, \mathbf{1}) \in B'$ and $(\hat{e}, \sigma), (\hat{e}, \tau) \in C'$. We need to show that $F((\hat{e}, \sigma)^{(\hat{e},\tau)}) = F(\hat{e}, \sigma)^{(\hat{e},\tau)}$. We begin by noting that multiplication in $\mathbf{D}$ *Wr* $\mathbf{C}$ is given by $(f_1, \sigma_1)(f_2, \sigma_2) = (f_1 \cdot (f_2 \circ \sigma_1^{-1}), \sigma_1 \circ \sigma_2)$. Next observe that $(\hat{e}, \sigma)^{(\hat{e},\tau)} = (\hat{e}, \tau^{-1} \circ \sigma \circ \tau)$ and $(f, \mathbf{1})^{(\hat{e},\tau)} = (f \circ \tau, \mathbf{1})$. It follows readily from the last two formulas that

(a) $(f, \mathbf{1}) \in F((\hat{e}, \sigma)^{(\hat{e},\tau)})$ iff $\forall x \in X,\ \tau^{-1} \circ \sigma \circ \tau(x) = x \Rightarrow f(x) = e$; and
(b) $(f, \mathbf{1}) \in F(\hat{e}, \sigma)^{(\hat{e},\tau)}$ iff $\forall x \in X,\ \sigma(x) = x \Rightarrow f \circ \tau^{-1}(x) = e$.

Suppose now that $(f, \mathbf{1}) \in F((\hat{e}, \sigma)^{(\hat{e},\tau)})$ and let $\sigma(x) = x$, for some $x \in X$. Then $\tau^{-1} \circ \sigma \circ \tau(\tau^{-1}(x)) = \tau^{-1}(x)$, and hence $f(\tau^{-1}(x)) = e$, that is, $f \circ \tau^{-1}(x) = e$. This shows that $(f, \mathbf{1}) \in F(\hat{e}, \sigma)^{(\hat{e},\tau)}$. Conversely, suppose $(f, \mathbf{1}) \in F(\hat{e}, \sigma)^{(\hat{e},\tau)}$, and let $\tau^{-1} \circ \sigma \circ \tau(x) = x$. Then $\sigma(\tau(x)) = \tau(x)$ and hence $f \circ \tau^{-1}(\tau(x)) = e$, that is, $f(x) = e$. Thus, $(f, \mathbf{1}) \in F((\hat{e}, \sigma)^{(\hat{e},\tau)})$.

(vii) Let $(f, \mathbf{1}), (g, \mathbf{1}) \in B'$ and $(\hat{e}, \sigma) \in C'$. We need to show that $(f, \mathbf{1}) \backslash (g, \mathbf{1}) \in F(\hat{e}, \sigma)$ iff $(g, \mathbf{1})/(f, \mathbf{1})^{(\hat{e},\sigma)} \in F(\hat{e}, \sigma)$. In light of the formula $(f, \mathbf{1})^{(\hat{e},\tau)} = (f \circ \tau, \mathbf{1})$, this is equivalent to showing that $(f \backslash g, \mathbf{1}) \in F(\hat{e}, \sigma)$ iff $(g/(f \circ \sigma), \mathbf{1}) \in F(\hat{e}, \sigma)$. But this is immediate, since we have at each fix point x of σ, $(f \backslash g)(x) = e$ iff $f(x) \backslash g(x) = e$ iff $f(x) \leq g(x)$ iff $(f \circ \sigma)(x) \leq g(x)$ iff $g(x)/(f \circ \sigma)(x) = e$ iff $(g/f \circ \sigma)(x) = e$. ■

We note, for future reference, that the partial order of $\mathbf{D}$ *Wr* $\mathbf{C}$ induced by F can be described as follows:

$$(f, \sigma) \leq (g, \tau) \text{ iff}$$

$$\sigma \leq \tau \text{ and } (\forall x \in X)[\sigma^{-1}(x) = \tau^{-1}(x) \Rightarrow f(x) \leq g(x)] \tag{10.1}$$

We are particularly interested in determining when a wreath product $\mathbf{D}$ *Wr* $\mathbf{C}$ is an RL. The following result shows that if C is an ℓ-group of automorphisms of a chain – by Holland's embedding theorem [12], any ℓ-group is isomorphic to such an ℓ-group – and D is an IRL, then $\mathbf{D}$ *Wr* $\mathbf{C}$ is an RL.

THEOREM 10.2. *Let C be an ℓ-group of automorphisms of a chain X and let D be an IRL. Then $\mathbf{A} = \mathbf{D}$ Wr $\mathbf{C}$ is an RL.*

PROOF. We only need to show that the partial order (10.1) is a lattice order. To this end, let (f, σ) and (g, τ) be two elements of the wreath product. It is easy to verify that $(f, \sigma) \vee (g, \tau) = (h_\vee, \sigma \vee \tau)$ and $(f, \sigma) \wedge (g, \tau) = (h_\wedge, \sigma \wedge \tau)$ where, for all $x \in X$,

$$h_\vee(x) = \begin{cases} f(x) & if \quad \sigma^{-1}(x) < \tau^{-1}(x) \\ g(x) & if \quad \tau^{-1}(x) < \sigma^{-1}(x) \\ f(x) \vee g(x) & if \quad \sigma^{-1}(x) = \tau^{-1}(x) \end{cases}$$

and

$$h_\wedge(x) = \begin{cases} f(x) & if \quad \sigma^{-1}(x) > \tau^{-1}(x) \\ g(x) & if \quad \tau^{-1}(x) > \sigma^{-1}(x) \\ f(x) \wedge g(x) & if \quad \sigma^{-1}(x) = \tau^{-1}(x). \end{cases}$$

■

The preceding two theorems immediately imply the last result of this paper.

COROLLARY 10.3. *$\mathcal{IRP} \times \mathcal{PG}$ is a proper subclass of $\mathcal{IRP} \times_s \mathcal{PG}$ and $\mathcal{IRL} \times \mathcal{LG}$ is a proper subvariety of $\mathcal{IRL} \times_s \mathcal{LG}$.*

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