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A Concrete Realization of the Hoare Powerdomain

Abstract. The lattice of non-empty Scott-closed subsets of a domain D is called the *Hoare powerdomain* of D. The Hoare powerdomain is used in programming semantics as a model for angelic nondeterminism. In this paper, we show that the Hoare powerdomain of any domain can be realized as the lattice of full subinformation systems of the domain's corresponding information system as well as the lattice of non-empty down-sets of the system's consistency predicate.

1. Introduction

In the late 1960's, Dana Scott introduced continuous lattices (see, Scott [13, 14, 15, 22]) into computer science as a means of providing mathematical models for a system of types that justify recursive definitions of these types. In time, the order theoretic models Scott and others considered evolved into what we now call domains (see Section 2).

The level of abstraction required to understand domain theory remained an obstacle to its widespread use. To remedy this problem, Scott imported from logic the notion of an *information system* to provide a set-theoretic approach to domains (see Scott [17]). In this setting, every information system gives rise to a domain in a canonical way (see Section 2 for details).

The Hoare powerdomain is an order-theoretic analog of the power set and is used in programming semantics as a model for angelic nondeterminism (see, for example, Plotkin [11]). The main result of this paper, Theorem 3.8, asserts that the Hoare powerdomain of any domain can be realized as the lattice of non-empty down-sets of the consistency predicate for the domain's corresponding information system. A corollary result – Corollary 4.8 – asserts that the Hoare powerdomain can be realized as the lattice of full subinformation systems of the domain's corresponding information system. These results provide concrete representations of the Hoare powerdomain by means of the corresponding information system.

2. Domains and Information Systems

Speaking loosely, a *domain* for a programming language is the underlying set of data objects for an admissible type equipped with an informationbased partial ordering. Over the years, a number of poset structures have been introduced to accommodate the needs of information theorists; and, at various times, all have come under the "domain" label (see for example Abramsky and Jung [1], Gunter [5], [6], [7], or Jung [9]). Today, the general consensus is that domains constitute some subcategory of algebraic posets with directed join homomorphisms (Scott-continuous maps) and particularly those subcategories which are cartesian closed (see Smyth [19] or Jung [8]). Our use of the term "domain" follows that of Davey and Priestley [2]; we pause briefly to explain.

A subset D of a poset P is *directed* if every finite subset of D has an upper bound in D. (Note that the empty subset of P is not directed.) A poset P is said to be *directed-complete*, or a DCPO for short, if the join of every directed subset of P exists in P. A subset I of poset $(P, \leq)$ is a *down-set* of P provided $I = {\downarrow}I = \{p \in P : \exists\ a \in I, p \leq a\}$. A down-set of a DCPO P is *Scott-closed* if it contains the join of each of its directed subsets. An element x of a DCPO P is *compact* if, whenever x is below the supremum of a directed set $D \subseteq P$, then $x \in {\downarrow}D$. We use $K(P)$ to denote the subposet of compact elements of P. A DCPO P is *algebraic* if, for all $p \in P$, the set $K(p) = {\downarrow}p \cap K(P)$ is directed and $p = \bigvee K(p)$. In this paper, we will use the term "domain" for an algebraic poset in which the meet of every non-empty subset exists. Equivalently, it is an algebraic poset in which the join of every upper bounded subset exists. Note that a domain has a least element.

We will let $\Gamma(P)$ denote the set of all Scott-closed subsets of a DCPO P, ordered by set-inclusion. It is easy to see that $\Gamma(P)$ is closed with respect to finite set-unions and arbitrary set-intersections; hence $\Gamma(P)$ is the family of closed sets for a topology on P, called the *Scott topology* on P. The lattice $\Gamma^*(D)$ of *non-empty* Scott-closed subsets of a domain D is called the *Hoare powerdomain* of D. The interested reader may wish to know that the Hoare powerdomain was originally described by Plotkin [12] as the ideal completion of the family of finite, non-empty sets of compact elements of the parent domain under the preorder

$$X \sqsubseteq Y \iff (\forall x \in X)(\exists y \in Y)(y \leq x)$$

The description of the Hoare powerdomain in terms of Scott-closed sets is due to Smyth [12].

We next turn our attention to information systems. Viewed from a logician's perspective, an information system for an object or a process is a triple $(S, \mathrm{Con}, \vdash)$, where S is a collection of propositions (or instructions) concerning the object or process, Con is a collection of finite subsets of S, which are somehow "consistent" with one another, and $\vdash$ is a relation of interdependence between members of Con. The members of S are thought of as providing simple bits of information about the object or process and are therefore called *tokens*. The set Con is called the *consistency predicate*, and $\vdash$ is known as a relation of *entailment*. An information system is assumed to obey certain common sense properties normally associated with the notions of consistency and entailment. These properties are made mathematically precise in the following definition. (In this definition and all the work that follows, we let $\mathrm{Fin}(S)$ denote the set of all finite subsets of a set S.)

DEFINITION 2.1. An *information system* is a triple $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ consisting of

(1) a set S whose elements are called *propositions* or *tokens*;

(2) a non-empty subset Con of $\mathrm{Fin}(S)$, called the *consistency predicate*; and

(3) a binary relation $\vdash$ on Con, called the *entailment relation*.

These entities satisfy the following axioms:

IS1 Con is a down-set of $\mathrm{Fin}(S)$ – with respect to set-inclusion – such that $\bigcup \mathrm{Con} = S$;

IS2 if $A \in \mathrm{Con}$ and $B \subseteq A$, then $A \vdash B$;

IS3 if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $B \vdash C$, then $A \vdash C$; and

IS4 if $A, B, C \in \mathrm{Con}$, $A \vdash B$, and $A \vdash C$, then $B \cup C \in \mathrm{Con}$ and $A \vdash (B \cup C)$.

Note that axiom IS1 implies that every singleton subset of S is a member of Con and that whenever $A \in \mathrm{Con}$ and $B \subseteq A$, then $B \in \mathrm{Con}$. Furthermore, axioms IS2 and IS3 imply that $(\mathrm{Con}, \vdash)$ is a preordered set, that is, $\vdash$ is a reflexive and transitive relation on Con.

We advise the reader that our definition of an information system is stated differently from the one commonly appearing in the literature (Scott [17] or Davey and Priestly [2]; see, however, Droste and Göbel [3]). In particular, the first two references define the entailment relation as a relation on the set Con $\times$ S. A comparison quickly shows our definition to be equivalent; it has the advantage of allowing us to think of $(\mathrm{Con}; \vdash)$ as a preordered set.

We close this section by describing the aforementioned correspondence between domains and information systems. Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. A subset X of $(\mathrm{Con}, \vdash)$ is a down-set of Con provided $X = \downarrow X = \{F \in \mathrm{Con} : \exists\, G \in X, G \vdash F\}$. A subset A of S is called an *element* of $\mathcal{S}$ provided $\mathrm{Fin}(A)$ is a down-set of $(\mathrm{Con}, \vdash)$. We let $\mathtt{El}(\mathcal{S})$ denote the set of "elements" of $\mathcal{S}$, ordered by set-inclusion. Routine application of the information system axioms shows the union of any directed subset of $\mathtt{El}(\mathcal{S})$ is a member of $\mathtt{El}(\mathcal{S})$. Thus, $\mathtt{El}(\mathcal{S})$ is a DCPO with respect to set-union. The set $\bar{\emptyset} = \{s \in S : \emptyset \vdash \{s\}\}$ is the least element of $\mathtt{El}(\mathcal{S})$; consequently, it is also routine to prove that $\mathtt{El}(\mathcal{S})$ is closed under non-empty intersections. We next claim that $\mathtt{El}(\mathcal{S})$ is an algebraic poset. Indeed, for each $F \in \mathrm{Con}$, $\bar{F} = \bigcup \ \downarrow F = \{s \in S : F \vdash \{s\}\}$ is the smallest member of $\mathtt{El}(\mathcal{S})$ containing F, as well as a compact member of $\mathtt{El}(\mathcal{S})$. Thus, $K(\mathtt{El}(\mathcal{S})) = \{\bar{F} : F \in \mathrm{Con}\}$ is the subposet of compact elements of $\mathtt{El}(\mathcal{S})$. Further, for each $A \in \mathtt{El}(\mathcal{S})$, $K(\mathtt{El}(\mathcal{S})) \cap \downarrow A = \{\bar{F} : F \in \mathtt{Fin}(A)\}$ is directed and $A = \bigcup\{\bar{F} : F \in \mathtt{Fin}(A)\}$. We have established that $\mathtt{El}(\mathcal{S})$ is a domain.

3. A Novel Representation of the Hoare Powerdomain

In this section, we establish the main result of this paper, Theorem 3.8, which states that if $\mathcal{S}$ is an information system, then the Hoare powerdomain of the domain associated with $\mathcal{S}$ is isomorphic to the lattice of non-empty down-sets of the consistency predicate of $\mathcal{S}$.

Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. By definition, the empty set $\emptyset \in \mathrm{Con}$; consequently, the intersection of any family of non-empty down-sets of $(\mathrm{Con}, \vdash)$ will always exceed $\downarrow\emptyset$ and therefore is never empty. The simple proof of the next result is left to the reader.

Lemma 3.1. *Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system.*

(1) *For all $F, G \in \mathrm{Con}$, $\bar{F} \subseteq \bar{G}$ if and only if $G \vdash F$. In particular, $\bar{F} = \bar{G}$ if and only if $G \vdash F$ and $F \vdash G$.*

(2) *$X \subseteq \mathrm{Con}$ is directed in $(\mathrm{Con}, \vdash)$ if and only if $\widetilde{X} = \{\bar{F} : F \in X\}$ is directed in $\mathtt{El}(\mathcal{S})$.*

It will be helpful to clarify what we mean by a "directed subset" in the statement of the preceding lemma: $X \subseteq \mathrm{Con}$ is directed in $(\mathrm{Con}, \vdash)$ provided every finite subset Y of X has an upper bound in X with respect to $\vdash$; that is, there exists $F \in X$ such that $F \vdash G$, for all $G \in Y$. Note that $\emptyset$ is not a directed subset of Con.

Definition 3.2. Let $\mathcal{S} = (\mathcal{S}, \mathrm{Con}, \vdash)$ be an information system. For non-empty $X \subseteq \mathrm{Con}$, let D_X denote the family of all directed subsets of X in $(\mathrm{Con}, \vdash)$, and let $\sigma(X) = \{\bigcup\{\bar{F} : F \in Y\} : Y \in D_X\}$.

Lemma 3.3. *Let* $\mathcal{S} = (\mathcal{S}, \mathrm{Con}, \vdash)$ *be an information system and let* $X \subseteq \mathrm{Con}$.

(1) *If* $X \in \mathrm{Con}$, *then* $\sigma(\{X\}) = \{\bar{X}\}$.

(2) $\sigma(X)$ *is a subset of* $\mathtt{El}(\mathcal{S})$.

(3) *If* X *is a down-set of* $(\mathrm{Con}, \vdash)$ *and* $F \in \mathrm{Con}$, *then* $F \in X$ *if and only if* $\bar{F} \in \sigma(X)$.

Proof. (1) This implication is immediate.

(2) Let $A \in \sigma(X)$. In light of Definition 3.2, there exists a directed $Y_A \subseteq X$ such that $A = \bigcup\{\bar{F} : F \in Y_A\}$. By Lemma 3.1(2), $\{\bar{F} : F \in Y_A\}$ is directed in $\mathtt{El}(\mathcal{S})$. Hence, A is a member of $\mathtt{El}(\mathcal{S})$, since the latter is a DCPO.

(3) Let X be a down-set of $(\mathrm{Con}, \vdash)$ and let $F \in \mathrm{Con}$. Definition 3.2 implies at once that $\bar{F} \in \sigma(X)$ whenever $F \in X$. Conversely, suppose that $\bar{F} \in \sigma(X)$. By Definition 3.2, there exists a directed subset $Y_{\bar{F}}$ of X such that $\bar{F} = \bigcup\{\bar{G} : G \in Y_{\bar{F}}\}$. Using the compactness of $\bar{F}$ in $\mathtt{El}(\mathcal{S})$, along with the fact that $\{\bar{G} : G \in Y_F\}$ is directed in $\mathtt{El}(\mathcal{S})$ by Lemma 3.1(2), we obtain that $\bar{F} = \bar{G}$, for some $G \in Y_F$. Invoking Lemma 3.1(1), we obtain successively $G \vdash F$ and $F \in X$, since X is a down-set of Con. ■

Lemma 3.4. *Let* $\mathcal{S}$ *be an information system. If* X *is a non-empty down-set of* Con, *then* $\sigma(X)$ *is a non-empty Scott-closed subset of* $\mathtt{El}(\mathcal{S})$.

Proof. Let X be a non-empty down-set of Con. It is clear that $\sigma(X)$ is non-empty. We first prove that $\sigma(X)$ is a down-set of $\mathtt{El}(\mathcal{S})$. Suppose that $A \in \sigma(X)$, $B \in \mathtt{El}(\mathcal{S})$, and $B \subseteq A$. We know that $A = \bigcup\{\bar{F} : F \in Y_A\}$, for some directed subset Y_A of X. We also have that $B = \bigcup\{\bar{G} : G \in \mathtt{Fin}(B)\}$. Since $\mathtt{Fin}(B)$ is clearly directed in $(\mathrm{Con}, \vdash)$, $\{\bar{G} : G \in \mathtt{Fin}(B)\}$ is directed in $\mathtt{El}(\mathcal{S})$ by Lemma 3.1(2). Thus, to prove that $B \in \sigma(X)$, it will suffice to prove that $\bar{G} \in \sigma(X)$ for each $G \in \mathtt{Fin}(B)$. Let $\bar{G}$ be such a compact element. The representation of A implies that $\bar{G} \subseteq \bar{F}$, for some $F \in Y_A$. Hence $F \vdash G$, by Lemma 3.1(1). Hence, we know $G \in X$ since X is a down-set of $(\mathrm{Con}, \vdash)$. We have shown that $\sigma(X)$ is a down-set of X.

Next, suppose that D is a directed subset of $\sigma(X)$. We need to prove that $\bigcup D \in \sigma(X)$. As above, it will suffice to prove that each compact element of $\mathtt{El}(\mathcal{S})$ below $\bigcup D$ belongs to $\sigma(X)$. To this end, let $\bar{G}$ be such an element, for $G \in \text{Con}$. The inclusion $\bar{G} \subseteq \bigcup D$ and the compactness of $\bar{G}$ imply that $\bar{G} \subseteq A$, for some $A \in D$. But then, $\bar{G} \in \sigma(X)$, since $\sigma(X)$ is a down-set of $\mathtt{El}(\mathcal{S})$. ■

DEFINITION 3.5. Let $\mathcal{S} = (S, \text{Con}, \vdash)$ be an information system and suppose that X is a subset of $\mathtt{El}(\mathcal{S})$. Let $\delta(X)$ be defined by

$$F \in \delta(X) \iff F \in \text{Con and } \bar{F} \in X.$$

LEMMA 3.6. *If $\mathcal{S} = (S, \text{Con}, \vdash)$ is an information system and X is a non-empty down-set of* Con*, then $X = \delta(\sigma(X))$.*

PROOF. Let X is a non-empty down-set of Con and let $F \in \text{Con}$. In view of Definition 3.5, $F \in \delta(\sigma(X)) \iff F \in \text{Con}$ and $\bar{F} \in \sigma(X)$. The second condition is equivalent to $F \in X$ by Lemma 3.3(3). This establishes the required equality. ■

LEMMA 3.7. *If $\mathcal{S}$ is an information system and X is a non-empty down-set of $\mathtt{El}(\mathcal{S})$, then $\delta(X)$ is a non-emty down-set of* Con*, and $X \subseteq \sigma(\delta(X))$. Furthermore, if X is Scott-closed, then $\sigma(\delta(X)) = X$.*

PROOF. Let $\mathcal{S}$ be an information system and let X be a non-empty down-set of $\mathtt{El}(\mathcal{S})$. To see that $\delta(X)$ is a down-set of $(\text{Con}; \vdash)$, suppose $A \in \delta(X)$, $B \in \text{Con}$, and $A \vdash B$. Axioms IS2 and IS4 imply that $B \subseteq \bar{A}$. Since $\bar{A} \in X$ and since X is a down-set of $\mathtt{El}(\mathcal{S})$, it follows that $\bar{B} \in X$. Hence, $B \in \delta(X)$.

Next, let $F \in \text{Con}$. We claim that

$$\bar{F} \in X \iff F \in \delta(X) \iff \bar{F} \in \sigma(\delta(X)) \tag{1}$$

Indeed, the first equivalence is the definition of δ (see Definition 3.5), while the second equivalence follows from Lemma 3.3 since, as we just proved, $\delta(X)$ is a down-set of Con.

An important consequence of (1) is that the compact elements of X and $\sigma(\delta(X))$ coincide. Note further that Lemma 3.4 implies that $\sigma(\delta(X))$ is a Scott-closed subset of $\mathtt{El}(\mathcal{S})$. These two facts, together with the standard representation, $A = \bigcup\{\bar{F} : F \in \mathtt{Fin}(A)\}$, of an element $A \in X$, immediately imply that $X \subseteq \sigma(\delta(X))$. Lastly, if X is Scott-closed, the same argument establishes the reverse inclusion. ■

Let $\Delta^*(\text{Con})$ be the lattice of non-empty down-sets of $(\text{Con}, \vdash)$. Recall that for any domain D, $\Gamma^*(D)$ denotes the Hoare powerdomain of D. The main result of this paper is an immediate consequence of the last two lemmas.

THEOREM 3.8. *Let $\mathcal{S}$ be an information system and let* $\texttt{El}(\mathcal{S})$ *be the domain of elements of* $\mathcal{S}$. *Then* $\Delta^*(\mathrm{Con})$ *is order isomorphic to* $\Gamma^*(\texttt{El}(\mathcal{S}))$.

PROOF. The assignments $\sigma : \Delta^*(\mathrm{Con}) \longrightarrow \Gamma^*(\texttt{El}(\mathcal{S}))$ and $\delta : \Gamma^*(\texttt{El}(\mathcal{S})) \longrightarrow \Delta^*(\mathrm{Con})$ defined above are mutually inverse by Lemmas 3.6 and 3.7. They are clearly isotone and therefore constitute an order isomorphism. ■

We remark that arbitrary meets and joins in $\Delta^*(\mathrm{Con})$ are simply set-intersection and union. Thus, the preceding representation immediately yields the well-known result – see, for example, Abramsky and Jung [1], Vickers[20], or Gierz *et al.* [4] – that the Hoare powerdomain of a domain is a bialgebraic (algebraic and dually algebraic) distributive lattice. The completely join-prime members of $\Delta^*(\mathrm{Con})$ are those of the form $\downarrow F = \{G \in \mathrm{Con} : F \vdash G\}$, for $F \in \mathrm{Con}$. (Recall that an element j of a complete lattice L is *completely join-prime* if, whenever j is below the join of an arbitrary $F \subseteq L$, then $j \in F$.)

4. Full Subinformation Systems

In this section, we consider substructures of an information system and derive a corollary result to Theorem 3.8. This result provides a concrete representation of the Hoare powerdomain as a substructure lattice of its parent information system. We begin by introducing these structures.

DEFINITION 4.1. Let $\mathcal{T} = (T, \mathrm{Con}_T, \vdash_T)$ and $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be information systems. $\mathcal{T}$ is a *subinformation system* of $\mathcal{S}$ provided $\mathcal{T}$ is an information system in its own right and $T \subseteq S$, $\mathrm{Con}_T \subseteq \mathrm{Con}$, and $\vdash_T \subseteq \vdash$.

It is easy to see that the binary relation $\sqsubseteq$, defined by $\mathcal{T} \sqsubseteq \mathcal{S}$ if and only if $\mathcal{T}$ is a subinformation system of $\mathcal{S}$, provides a partial order on the set of all subinformation systems of $\mathcal{S}$. We will let $\texttt{Sub}(\mathcal{S})$ denote the set of all subinformation systems of $\mathcal{S}$, partially ordered under $\sqsubseteq$.

DEFINITION 4.2. Let $\mathcal{T} = (T, \mathrm{Con}_T, \vdash_T)$ be a subinformation system of $\mathcal{S} = (S, \mathrm{Con}, \vdash)$. We say that $\mathcal{T}$ is a *full* subinformation system of $\mathcal{S}$ provided

(1) Con_T is a lower-set of Con (with respect to entailment); and

(2) for all $A, B \in \mathrm{Con}_T$, $A \vdash_T B \iff A \vdash B$.

It is routine to prove that if $\mathcal{T} = (\mathcal{T}, \mathrm{Con}_{\mathcal{T}}, \vdash_{\mathcal{T}})$ and $\mathcal{U} = (\mathcal{U}, \mathrm{Con}_{\mathcal{U}}, \vdash_{\mathcal{U}})$ are full subinformation systems of $\mathcal{S} = (\mathcal{S}, \mathrm{Con}, \vdash)$, then $\mathcal{T}$ is a full subinformation system of $\mathcal{U}$ if and only if $\mathrm{Con}_T \subseteq \mathrm{Con}_U$. We will use $\texttt{FSub}(\mathcal{S})$ to denote the subposet of $\mathtt{Sub}(\mathcal{S})$ consisting of all full subinformation systems of an information system $\mathcal{S}$. We note that the triple $(\emptyset; \{\emptyset\}; (\emptyset, \emptyset))$ is an information system and a subinformation system of every information system. We will refer to it as the *empty information system*.

DEFINITION 4.3. Let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system. For non-empty $X \subseteq \mathrm{Con}$, let $\mathtt{FSub}(X) = (\bigcup X, X, \vdash_X)$, where $\vdash_X$ denotes the restriction of $\vdash$ to X.

LEMMA 4.4. *If $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ is an information system and X a non-empty down-set of* $(\mathrm{Con}, \vdash)$*, then* $\mathtt{FSub}(X)$ *is a full subinformation system of $\mathcal{S}$.*

Given any information system $\mathcal{S}$, we observe that $\mathtt{FSub}(\downarrow\emptyset)$ is the least member of $\mathtt{FSub}(\mathcal{S})$, and that $\mathtt{FSub}(\downarrow\emptyset)$ is the empty information system if and only if $\downarrow\emptyset = \{\emptyset\}$.

The preceding result shows that each non-empty down-set X of $(\mathrm{Con}, \vdash)$ gives rise to a full-subinformation system of $\mathcal{S}$. On the other hand, if $\mathcal{T}$ is a full subinformation system of $\mathcal{S}$, then

$$\mathtt{Down}(\mathcal{T}) = \mathrm{Con}_{\mathcal{T}}$$

is a down-set of $(\mathrm{Con}, \vdash)$.

The straightforward proofs of the next two results are left to the reader.

LEMMA 4.5. *Let $\mathcal{S}$ be an information system and let X be a non-empty down-set of* Con*. Then* $\mathtt{Down}(\mathtt{FSub}(X)) = X$.

LEMMA 4.6. *Let $\mathcal{T}$ be a full subinformation system of an information system $\mathcal{S}$. Then* $\mathtt{FSub}(\mathtt{Down}(\mathcal{T})) = \mathcal{T}$.

In light of Lemmas 4.5 and 4.6, we have the following results.

THEOREM 4.7. *For any information system $\mathcal{S}$, the lattice* $\Delta^*(\mathrm{Con})$ *is order isomorphic to* $\mathtt{FSub}(\mathcal{S})$.

COROLLARY 4.8. *For any information system $\mathcal{S}$, the Hoare powerdomain* $\Gamma^*(\mathtt{El}(\mathcal{S}))$ *is order isomorphic to* $\mathtt{FSub}(\mathcal{S})$.

Among other things, Theorem 4.7 and Corollary 4.8 tell us much about the structure of $\mathtt{FSub}(\mathcal{S})$. In particular, it is a bialgebraic distributive lattice whose completely join-prime members are precisely the substructures

$$\mathtt{FSub}(\downarrow A) = (\bigcup \downarrow A, \downarrow A, \vdash_{\downarrow A}),$$

where $A \in \mathrm{Con}, A \neq \emptyset$.

If we recall that meets and joins in $\Delta^*(\mathrm{Con})$ are simply set intersection and union, then we can also describe meets and joins in $\mathtt{FSub}(\mathcal{S})$ quite easily. Indeed, let $\mathcal{S} = (S, \mathrm{Con}, \vdash)$ be an information system and let $F = \{\mathcal{T}_i : i \in I\}$ be any family of full subinformation systems of $\mathcal{S}$, where each $\mathcal{T}_i = (T_i, \mathrm{Con}_i, \vdash_i)$. Then $\bigwedge \mathcal{F}$ is given by $\bigwedge \mathcal{F} = \mathcal{S}$ if $F = \emptyset$ and

$$\begin{aligned}\bigwedge \mathcal{F} &= \mathtt{FSub}(\mathtt{Down}(\bigcap\{\mathrm{Con}_i : i \in I\})) \\ &= (\bigcap\{T_i : i \in I\}, \bigcap\{\mathrm{Con}_i : i \in I\}, \bigcap\{\vdash_i : i \in I\})\end{aligned}$$

if $F \neq \emptyset$. Likewise, $\bigvee \mathcal{F}$ is given by $\bigvee \mathcal{F} = \mathtt{FSub}(\downarrow\emptyset)$ if $F = \emptyset$ and

$$\begin{aligned}\bigvee \mathcal{F} &= \mathtt{FSub}(\mathtt{Down}(\bigcup\{\mathrm{Con}_i : i \in I\})) \\ &= (\bigcup\{T_i : i \in I\}, \bigcup\{\mathrm{Con}_i : i \in I\}, \bigcup\{\vdash_i : i \in I\})\end{aligned}$$

if $F \neq \emptyset$.

We point out that the simple description of joins in $\mathtt{FSub}(\mathcal{S})$ fails in the complete lattice $\mathtt{Sub}(\mathcal{S})$.

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