F. Paoli
C. Tsinakis

On Birkhoff's Common Abstraction Problem

Abstract. In his milestone textbook *Lattice Theory*, Garrett Birkhoff challenged his readers to develop a "common abstraction" that includes Boolean algebras and lattice-ordered groups as special cases. In this paper, after reviewing the past attempts to solve the problem, we provide our own answer by selecting as common generalization of $\mathcal{BA}$ and $\mathcal{LG}$ their join $\mathcal{BA} \vee \mathcal{LG}$ in the lattice of subvarieties of $\mathcal{FL}$ (the variety of FL-algebras); we argue that such a solution is optimal under several respects and we give an explicit equational basis for $\mathcal{BA} \vee \mathcal{LG}$ relative to $\mathcal{FL}$. Finally, we prove a Holland-type representation theorem for a variety of FL-algebras containing $\mathcal{BA} \vee \mathcal{LG}$.

Keywords: Residuated lattice, FL-algebra, Substructural logics, Boolean algebra, Lattice-ordered group, Birkhoff's problem, History of 20$^{\text{th}}$ C. algebra.

1. Introduction

In his milestone textbook *Lattice Theory* [2, Problem 108], Garrett Birkhoff challenged his readers by suggesting the following project:

> Develop a common abstraction that includes Boolean algebras (rings) and lattice ordered groups as special cases.

Over the subsequent decades, several mathematicians tried their hands at Birkhoff's intriguing problem. Its very formulation, in fact, intrinsically seems to call for reiterated attempts: unlike most problems contained in the book, for which it is manifest what would count as a correct solution, this one is stated in sufficiently vague terms as to leave it open to debate whether any proposed answer is really adequate. It appears to us that Rama Rao puts things right when he remarks [28, p. 411]:

> A Boolean ring or an ℓ-group may be regarded as an algebra $\mathbf{A} = (A; \wedge, \vee, +, -)$ of species $(2, 2, 2, -1)$ and each of them is equationally definable; consequently any algebra of species $(2, 2, 2, -1)$ satisfying the identities common to Boolean rings

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> and ℓ-groups can be taken as a solution to Birkhoff's problem. However, such a common abstraction will be uninteresting unless it possesses as much as possible of the richness of the structures common to both ℓ-groups and Boolean rings.

In other words: although any class of structures that includes both Boolean algebras (or Boolean rings) and ℓ-groups could be considered as an answer to Birkhoff's problem, not all such generalizations stand on an equal footing. Some common abstractions are better than others in that they preserve a greater share of the properties enjoyed by both classes. The more common properties survive in the abstraction, the more acceptable it will be as a realization of Birkhoff's project.

A question, now, arises in a natural way: is there any possible solution we could count as *optimal*? To answer this question, we need to be somewhat more specific than Birkhoff or Rama Rao as to what we mean by "common property." Hereafter we summarize our — necessarily biased — viewpoint on the issue by listing a few *desiderata*.

- D1 As regards the *general* properties, the common abstraction should at the very least have the same *class type* as Boolean algebras and ℓ-groups: since both classes are varieties of total algebras, the generalization should also be such.
- D2 As regards the *theoretical* properties, an indispensable requirement is that the *equational theory* of the common abstraction be as large as it can: namely, one should formulate the common abstraction $\mathcal{V}$ in a given signature ν such that term equivalent versions of $\mathcal{BA}$ (the variety of Boolean algebras) and $\mathcal{LG}$ (the variety of ℓ-groups) are varieties of signature ν, and the equational theory of $\mathcal{V}$ is the *intersection* of the equational theories of $\mathcal{BA}$ and $\mathcal{LG}$.
- D3 As regards the *metatheoretical* properties, it would be desirable that $\mathcal{V}$ share with $\mathcal{BA}$ and $\mathcal{LG}$ as many as possible of the nice universal algebraic properties characterizing them: to name just a few, significant properties of congruence lattices (like congruence distributivity, congruence permutability, or congruence regularity) or the availability of a satisfactory representation.
- D4 Finally, it would be definitely preferable if the proposed solution were not *ad hoc* — meaning that it would serve other purposes than the present one or, at least, it would be a subvariety of some well-known, significant and independently motivated variety.

The remainder of the paper is divided as follows. In Section 2 we review the past attempts to solve the problem, showing that all of them — at least all the ones with which we are familiar — fail one or another of D1-D4. In Section 3, we recall some basic notions about *residuated lattices* and *FL-algebras*. In Sections 4 and 5, we provide our own answer to the problem by selecting as common generalization of $\mathcal{BA}$ and $\mathcal{LG}$ their join $\mathcal{BA} \vee \mathcal{LG}$ in the lattice of subvarieties of $\mathcal{FL}$ (the variety of FL-algebras); we also give an explicit equational basis for $\mathcal{BA} \vee \mathcal{LG}$ relative to $\mathcal{FL}$. Our solution, which is based on some results in [19] and [11], automatically satisfies desiderata D1, D2, and D4. Results bearing on the issue of D3, such as a Holland-type representation theorem for a variety of FL-algebras containing $\mathcal{BA} \vee \mathcal{LG}$, are reserved for the final section.

2. A survey of previous attempts

2.1. D. Ellis (1953)

Historically, D. Ellis [9] made the first attempt to solve Birkhoff's problem in the second of his "Notes on the foundation of lattice theory". He was interested in characterizing such properties of lattices as distributivity or modularity in terms of the existence of special binary operations named *c-functions*:

Definition 1. Let $\mathbf{L}$ be a lattice. A binary operation $*$ on $\mathbf{L}$ is a *c-function* on L provided for all $a, b \in L$:

$$a * b = (a \wedge b) * (a \vee b)\,.$$

Ellis shows, for example, that a lattice is distributive if and only if it admits a c-function enjoying some kind of cancellation property. Since both Boolean algebras and *Abelian* ℓ-groups can be viewed as lattices with c-functions (in the former case $a * b$ is the symmetric difference $(a \wedge \neg b) \vee (b \wedge \neg a)$, while in the latter the role of a c-function is played by group multiplication), Ellis observes, these examples "suggest [lattices with c-functions] as a possible partial answer to Birkhoff's problem" [9, p. 257]. The author's cautious statement exposes his awareness that the failure to encompass all ℓ-groups as instances of lattices with c -functions makes his suggestion questionable.

2.2. K. L. N. Swamy (1965-1966)

K. L. N. Swamy seems to be the first author who gets reasonably close to the heart of the matter by observing that both Boolean algebras and ℓ-groups are (dually) *residuated* structures. Actually, the problem Swamy wants to tackle in a series of papers in *Mathematische Annalen* in the mid-sixties [32, 33, 34] is at the same time more general and more specific than Birkhoff's one: he aims at developing a common generalization of Brouwerian (dual Heyting) algebras and Abelian ℓ-groups. The notion Swamy deems appropriate for this purpose is defined below.

Definition 2. A *dually residuated lattice-ordered semigroup*, for short DRL-semigroup, is an algebra $\mathbf{A} = (A, +, -, \wedge, \vee, 0)$ of type $(2, 2, 2, 2, 0)$ such that:

(1) the reduct $(A, +, \wedge, \vee, 0)$ is an Abelian ℓ-monoid;

(2) for all $a, b \in A$, $a - b = \min\{x \in A : a \leq b + x\}$: in other words, subtraction is the coresidual of addition; and

(3) for all $a, b \in A$, $((a - b) \vee 0) + b \leq a \vee b$ and $0 \leq a - a$.

Swamy points out that Abelian ℓ-groups are DRL semigroups under the obvious matching of fundamental operations, while Boolean algebras are DRL semigroups by taking lattice join as addition and difference $(a \wedge \neg b)$ as subtraction; he also shows that DRL semigroups form a variety, and provides equational bases for the subvarieties of Boolean algebras and Abelian ℓ-groups. Finally, he specifies necessary and sufficient conditions for a DRL semigroup to be decomposed as a direct product of a Boolean algebra and an Abelian ℓ-group.

It is only in the third paper of the series that Swamy introduces a noncommutative generalization of DRL-semigroups, which is evidently far more interesting from the viewpoint of the present paper — and also in itself: if right coresiduals are added, the resulting variety is term equivalent to the variety of *GBL algebras*, introduced in [18] and deeply investigated by authors working on residuated lattices. Unfortunately, no results are proved on that score: rather, the reader is referred to the author's PhD thesis which — as far as we could ascertain — never appeared in print. Swamy's neglected work has been taken up again only in relatively recent times (see e.g. [20, 22]), where DRL semigroups in their noncommutative incarnation are studied and renamed, more appropriately, *DRL monoids*. However, the language of residuated lattices has gained favor among algebraic logicians, and the transition from dually residuated structures to residuated structures, and vice versa, is obvious — one simply has to reverse the underlying partial order.

2.3. O. Wyler (1966)

The main motivational spur for a number of authors who have worked on Birkhoff's problem is given by their desire to find a general unified framework for the theories of measure and integration. Putting off a less cursory discussion on this subject until we survey the work by K. D. Schmidt [29, 30], we observe that O. Wyler [35] falls within this category. Recall that a *clan of functions* is a sublattice of the vector lattice of all real-valued functions on some space X that: a) contains $g-f$ whenever it contains f, g and $f \leq g$; and b) contains $h-f$ whenever it contains $f, g, h, g-f$, and $h-g$. Trying to achieve an abstract counterpart of this concept, Wyler introduces a class of partial algebras called *abstract clans*:

Definition 3. An *abstract clan* is a lattice $\mathbf{L}$ with a partial binary operation $-$ (called *subtraction*), defined on $\Sigma \subseteq L^2$, that satisfies the following conditions for all $a, b, c \in L$:

(1) If $a \leq b$, then $(a, b) \in \Sigma$;
(2) If $a \vee b \leq c$, then $a \leq b$ iff $c-b \leq c-a$;
(3) If $(b, a), (c, a) \in \Sigma$ and $b \leq c$, then $b-a \leq c-a$ and $(c-a)-(b-a) = c-b$;
(4) If $b \leq c$ and $a \leq c-b$, then there is $d \in L$ such that $(c, d) \in \Sigma$ and $c-d = (c-b)-a$.

Condition (4) in Definition 3 can be interpreted as postulating the existence of a partial addition on the clan. A clan is *commutative* if so is such an addition, and *total* if $\Sigma = L^2$. Lattice-ordered groups are total clans under both division operations: $a-b$ can be defined either as ab^{-1} or as $b^{-1}a$. Boolean algebras, on the other hand, are commutative clans under $\Sigma = \{(a, b) \in L^2 : b \leq a\}$ and $a-b = a \wedge \neg b$.

2.4. T. Nakano (1967)

T. Nakano [26, p. 355] is intrigued by the "various parallelisms between the theory of rings and that of partly ordered systems," quoting as an example the similarities in the proofs of mutually independent theorems like Lorenzen's characterization of representable ℓ-groups and the representation of integral domains as intersections of quasilocal integral domains. Nakano surmises that "there should be an underlying theorem generalizing both." This leads him towards a ring-like generalization of ℓ-groups that also includes Boolean rings.

Whereas Wyler's addition was partial, Nakano's one is everywhere defined, but *multivalued*. We give a simple example. Let $\mathbf{A} = (A, +, -, 0)$ be an Abelian group, and let $M_{\mathbf{A}} = \{\{a, -a\} : a \in A\}$. We refer to each class $\{a, -a\}$ in this subsection by the symbol $\overline{a}$. We can now define a multivalued operation $\boxplus : (M_{\mathbf{A}})^2 \to \wp(M_{\mathbf{A}})$ in such a way that, for any $\overline{a}, \overline{b} \in M_{\mathbf{A}}$,

$$\overline{a} \boxplus \overline{b} = \left\{\overline{a+b}, \overline{a-b}\right\}.$$

Abstracting away from this particular example, Nakano introduces multimodules and multirings as follows:

Definition 4. A *multimodule* is a set M together with an operation $\boxplus$: $M \to \wp(M)$ such that, if we define $A \boxplus B = \bigcup\{a \boxplus b : a \in A, b \in B\}$, the following conditions are satisfied for any $a, b, c \in M$:

(1) $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c$

(2) $a \boxplus b = b \boxplus a$

(3) $c \in a \boxplus b$ implies $a \in b \boxplus c$

A *multiring* is a multimodule together with a second operation $\boxdot : M \to M$ such that, if we define $A \boxdot B = \{a \boxdot b : a \in A, b \in B\}$, the following conditions are satisfied for any $a, b, c \in M$:

(4) $a \boxdot (b \boxdot c) = (a \boxdot b) \boxdot c$

(5) $a \boxdot (b \boxplus c) = (a \boxdot b) \boxplus (a \boxdot c)$

(6) $(b \boxplus c) \boxdot a = (b \boxdot a) \boxplus (c \boxdot a)$

It is easily seen that Boolean algebras are multirings under $a \boxdot b = a \vee b$ and $a \boxplus b = \{c : a \wedge b = b \wedge c = c \wedge a\}$, while ℓ-groups are multirings under $a \boxdot b = ab$ and the same addition operation as Boolean algebras. A significant common generalization of theorems about commutative rings and ℓ-groups is not really attained in this paper, but some partial results in this direction are presented.

2.5. V. V. Rama Rao (1969)

Rama Rao was an "academic brother" of Swamy, for both were students of Subrahmanyam at Andhra University in Waltair, India. His approach to the problem in [28] is, indeed, not too dissimilar from Swamy's: he considers the variety of all algebras of type $(2, 2, 2, 2, 0)$ and seeks necessary and sufficient conditions for an algebra of that type to be representable as a direct product of a Boolean ring and of an ℓ-group. His main theorem in the paper is a characterization of the class of algebras having this property by means of a list of axioms which, unfortunately, is not wholly equational. Although the fact that the class is a variety can be recovered from general results in universal algebra (see Section 4 below), the paper falls short of providing an equational basis for that variety.

2.6. K. D. Schmidt (1985-1988)

Let $\mathbf{R}$ be a ring of subsets over some nonempty set X, and let $\mathbf{V}$ be a vector space. Recall that a ($\mathbf{V}$-valued) *vector measure* on $\mathbf{R}$ is a map $\varphi : R \to V$ such that $\varphi(A+B) = \varphi(A) + \varphi(B)$ holds for disjoint $A, B \in R$. Recall, moreover, that the span of the characteristic functions of sets in R (taken in the space of all real-valued functions over X) is the universe of a vector lattice $\mathbf{D}(\mathbf{R})$ under pointwise defined operations, and that there is a bijective correspondence between $\mathbf{V}$-valued vector measures on $\mathbf{R}$ and linear transformations from $\mathbf{D}(\mathbf{R})$ to $\mathbf{V}$. Schmidt [31, p. 138] observes:

> This one-one correspondence can be used to obtain results on vector measures from corresponding ones on linear operators — provided that suitable results on linear operators are known. Instead of reducing problems on vector measures to those on linear operators, one can try to develop a common approach to vector measures on an algebra of sets and linear operators on a vector lattice.

This common approach is first attempted by simultaneously generalizing Boolean rings and Abelian ℓ-groups. In [29], Schmidt introduces lattice ordered partial semigroups as follows:

Definition 5. A *lattice ordered partial semigroup* is a lattice $(L, \wedge, \vee)$ with a partial operation $+$, defined on $\Sigma \subseteq L^2$, which is associative, commutative, compatible with lattice order (provided the relevant sums are defined) and has an identity element 0. A lattice ordered partial semigroup has the *cancellation property* if it is cancellative as a partial semigroup, while it has the *difference property* if for any $a, b \in L$ there exists a $c \in L$ such that $(a, c), (a \wedge b, c) \in \Sigma$, $a + c = a \vee b$ and $(a \wedge b) + c = b$.

It can be checked that any Boolean algebra (or ring) $\mathbf{B}$ can be turned into a lattice ordered partial semigroup by taking the set $\left\{(a, b) \in B^2 : a \wedge b = 0\right\}$ as Σ and lattice join as addition. Likewise, any Abelian ℓ-group $\mathbf{G}$ can be viewed as a lattice ordered partial semigroup if we take $\Sigma = G^2$ and group multiplication as addition. In the same paper, necessary and sufficient conditions are given for a lattice ordered partial semigroup to be a Boolean algebra or an Abelian ℓ-group.

In a subsequent paper [30], lattice ordered partial semigroups with the cancellation property and the difference property are generalised to the non-commutative case under the name of *minimal clans*. Schmidt observes that his minimal clans are equivalent to a proper subclass of Wyler's abstract clans while still including Boolean algebras and ℓ-groups as examples, whence his generalization is more effective. It is interesting to remark that minimal clans are referred to by some authors (see e.g., [6, 8, 15]) as *Vitali spaces*.

2.7. E. Casari (1989)

While the motivational stimuli for the classes of structures we reviewed so far arise from different areas of mathematics, Casari [4] was driven to Birkhoff's problem by linguistical and philosophical considerations. He was interested in giving an account of such natural language comparative sentences as "c is at most as P as d is Q," where c, d are names and P, Q are predicates. If we accept that sentences may admit of different "degrees of truth" (a widely shared concept in the area of many-valued logics), the aforementioned sentence can be considered true when "c is P" is at most as true as "d is Q." To formalize his idea, Casari needed an implication connective which comes out true exactly when its antecedent is at most as true as its consequent. Although fuzzy logics share the same basic assumptions, for a number of reasons[1] Casari was dissatisfied with such an approach.

Along these lines, Casari came to build up a logical system for *comparative logic* whose algebraic models are called *Abelian ℓ-pregroups*. The variety of Abelian ℓ-pregroups is term equivalent to the commutative subvariety of $\mathcal{InFL}$ (see Section 3 below) satisfying the equations

P1 $x \to x \approx 1$

P2 $0 \to 1 \approx 1$

Abelian ℓ-groups are exactly the Abelian ℓ-pregroups satisfying the equation $0 \approx 1$, while Boolean algebras are the Abelian ℓ-pregroups satisfying $xy \approx x \wedge y$.

[1] For example, the use of bounded algebras as systems of truth degrees in mainstream fuzzy logics prevents a proper treatment of comparative sentences of the form "c is less P than d" when both c and d are clear-cut instances of P, yet it makes sense to say that d is more P than c is.

The theory of Abelian ℓ-pregroups was investigated in subsequent years in ways that are relevant for Birkhoff's problem. Minari (circa 1992) generalized the theory of Abelian ℓ-pregroups to the noncommutative case, but unfortunately his notes never made their way to the publisher. One of the present authors [27], on the other hand, stayed commutative and proved that the join of the varieties of Abelian ℓ-groups and of Boolean algebras in the subvariety lattice of the variety of Abelian ℓ-pregroups is term equivalent to the variety whose equational basis relative to the commutative subvariety of $\mathcal{I}n\mathcal{FL}$ is given by P1, P2 and the following equations:

P3 $x\,((x \to y) \wedge 1) \approx x \wedge y$

P4 $x\,(y \wedge z) \approx xy \wedge xz$

P5 $x \to ((y \to 0) \to y) \approx (x \to y)\,(0 \to y)$

2.8. N. Galatos and C. Tsinakis (2005)

So far, the most successful common generalization of Boolean algebras and ℓ-groups is undoubtedly represented by *residuated lattices* (see below, and also [10]). In [11], Galatos and one of the present authors consider a variety of residuated lattices that simultaneously generalizes Chang's *MV-algebras* (one of the most important class of structures in many-valued logic: see e.g. [5]) to the non-commutative, unbounded and non-integral case. These algebras are called *generalized MV-algebras*, or *GMV-algebras*. The class $\mathcal{GMV}$ of all GMV algebras is broad enough to include all ℓ-groups and all Boolean algebras (which are nothing but idempotent MV algebras). One of the main results in this paper is a representation of any GMV-algebra as a direct product of an ℓ-group and a nucleus retraction of the negative cone of an ℓ-group. More details about $\mathcal{GMV}$ will be given in the remainder of the present paper.

3. Basic facts about residuated lattices

We refer the reader to [3, 18, 25] or [10] for basic results in the theory of residuated lattices. Here, we only review background material needed in the remainder of the paper.

A binary operation $\cdot$ on a partially ordered set $(A, \leq)$ is said to be *residuated* provided there exist binary operations $\backslash$ and $/$ on A such that for all $a, b, c \in A$,

$$\text{(Res)}\ a \cdot b \leq c \ \text{ iff } \ a \leq c/b \ \text{ iff } \ b \leq a\backslash c.$$

We refer to the operations $\backslash$ and $/$ as the *left residual* and *right residual* of $\cdot$, respectively. As usual, we write xy for $x \cdot y$, x^2 for xx and adopt the convention that, in the absence of parentheses, $\cdot$ is performed first, followed by $\backslash$ and $/$, and finally by $\vee$ and $\wedge$.

The residuals may be viewed as generalized division operations, with x/y being read as "x over y" and $y\backslash x$ as "y under x". In either case, x is considered the *numerator* and y is the *denominator*. They can also be viewed as generalized implication operators, with x/y being read as "x if y" and $y\backslash x$ as "if y then x". In either case, x is considered the *consequent* and y is the *antecedent*. We tend to favor $\backslash$ in calculations, but any statement about residuated structures has a "mirror image" obtained by reading terms backwards (i.e., replacing $x \cdot y$ by $y \cdot x$ and interchanging x/y with $y\backslash x$).

We are primarily interested in the situation where $\cdot$ is a monoid operation with unit element 1 and the partial order $\leq$ is a lattice order. In this case, we add the monoid unit and the lattice operations to the similarity type and refer to the resulting structure $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ as a *residuated lattice*. An *FL-algebra* is an algebra $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1, 0)$ such that the reduct $(A, \vee, \wedge, \cdot, \backslash, /, 1)$ is a residuated lattice; in other words, we do not assume anything about the additional constant 0. Throughout this paper, the class of residuated lattices will be denoted by $\mathcal{RL}$ and that of FL-algebras by $\mathcal{FL}$. We adopt the convention that when a class is denoted by a string of calligraphic letters, then the members of that class will be referred to by the corresponding string of Roman letters. Thus an RL is a residuated lattice, and an FL is an FL-algebra.

The existence of residuals has the following basic consequences, which will be used in the remainder of the paper without explicit reference.

PROPOSITION 6. *Let* $\mathbf{A}$ *be an RL.*

(1) *The multiplication preserves all existing joins in each argument; i.e., if* $\bigvee X$ *and* $\bigvee Y$ *exist for* $X, Y \subseteq A$, *then* $\bigvee_{x \in X, y \in Y}(xy)$ *exists and*

$$\Big(\bigvee X\Big)\Big(\bigvee Y\Big) = \bigvee_{x \in X, y \in Y}(xy).$$

(2) *The residuals preserve all existing meets in the numerator, and convert existing joins to meets in the denominator, i.e. if* $\bigvee X$ *and* $\bigwedge Y$ *exist for* $X, Y \subseteq A$, *then for any* $z \in A$, $\bigwedge_{x \in X}(x\backslash z)$ *and* $\bigwedge_{y \in Y}(z\backslash y)$ *exist and*

$$\Big(\bigvee X\Big)\Big\backslash z = \bigwedge_{x \in X}(x\backslash z) \quad \text{and} \quad z\Big\backslash\Big(\bigwedge Y\Big) = \bigwedge_{y \in Y}(z\backslash y).$$

(3) *The following identities and quasi-identities (and their mirror images) hold in* $\mathbf{A}$.

(a) $(x\backslash y)z \leq x\backslash yz$

(b) $1\backslash x \approx x$

(c) $x\backslash y \leq zx\backslash zy$

(d) $x \leq y \Rightarrow y\backslash z \leq x\backslash z$

(e) $(x\backslash y)(y\backslash z) \leq x\backslash z$

(f) $xy\backslash z \approx y\backslash(x\backslash z)$

(g) $x\,(x\backslash 1)\,\backslash 1 \approx 1$

(h) $x\backslash(y/z) \approx (x\backslash y)/z$

Proposition 7. *$\mathcal{RL}$ and $\mathcal{FL}$ are finitely based varieties in their respective signatures, for the residuation conditions (Res) can be replaced by the following equations (and their mirror images):*

(i) $y \leq x\backslash\,(xy \vee z)$

(ii) $x\,(y \vee z) \approx xy \vee xz$

(iii) $y\,(y\backslash x) \leq x$

Given an RL $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1)$ or an FL $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1, 0)$, an element $a \in A$ is said to be *integral* if $1/a = 1 = a\backslash 1$, and $\mathbf{A}$ itself is said to be *integral* if every member of A is integral. We denote by $\mathcal{IRL}$ the variety of all integral RLs, and by $\mathcal{FL}_{\mathbf{i}}$ (following [10]) the variety of all integral FLs. $\mathcal{FL}_{\mathbf{w}}$ will denote the variety of FLs whose equational basis relative to $\mathcal{FL}_{\mathbf{i}}$ is given by

E1 $0\backslash x \approx 1$

Boolean algebras are term equivalent to the subvariety of $\mathcal{FL}_{\mathbf{w}}$ whose equational basis relative to $\mathcal{FL}_{\mathbf{w}}$ is given by the equations

E2 $xy \approx x \wedge y$

E3 $x/\,(y\backslash x) \approx x \vee y \approx (x/y)\,\backslash x$

(cf. [18] or [10]), which we denote by $\mathcal{BA}$. Its members will be referred to as Boolean algebras.

An element $a \in A$ is said to be *invertible* if $(1/a)a = 1 = a(a\backslash 1)$. This is of course true if and only if a has a (two-sided) inverse a^{-1}, in which case $1/a = a^{-1} = a\backslash 1$. The RLs and the FLs in which every element is invertible are therefore precisely the ℓ-groups. Perhaps a word of caution is appropriate here. An *ℓ-group* is usually defined in the literature as an algebra $\mathbf{G} = (G, \wedge, \vee, \cdot, {}^{-1}, e)$ such that $(G, \wedge, \vee)$ is a lattice, $(G, \cdot, {}^{-1}, e)$ is a group, and multiplication is order preserving (or, equivalently, it distributes over the lattice operations). The variety of ℓ-groups is term equivalent to the subvariety of $\mathcal{RL}$ defined by the equations $(1/x)x \approx 1 \approx x(x\backslash 1)$; the term equivalence is given by $x^{-1} = 1/x$ and $x/y = xy^{-1}, x\backslash y = x^{-1}y$. It is also term equivalent to the subvariety of $\mathcal{FL}$ defined by the equations $(1/x)x \approx 1 \approx x(x\backslash 1)$ and $0 \approx 1$. With a slight abuse, we identify with each other the aforementioned subvarieties, which we denote by $\mathcal{LG}$. Its members will be referred to as *ℓ-groups*.

We already mentioned GMV algebras as simultaneous generalizations of MV algebras to the noncommutative, unbounded and nonintegral case. GMV algebras are RLs; more precisely, the variety $\mathcal{GMV}$ of GMV algebras is axiomatized relative to $\mathcal{RL}$ by the equations

E4 $\quad x/((x \vee y)\backslash x) \approx x \vee y \approx (x/(x \vee y))\backslash x.$

The same equations also axiomatize the linguistic expansion of $\mathcal{GMV}$ by the constant 0 – which we also call $\mathcal{GMV}$, with a slight terminological abuse – relative to $\mathcal{FL}$. The variety $\mathcal{IGMV}$ of integral GMV-algebras is axiomatized relative to $\mathcal{IRL}$ (or $\mathcal{FL}_{\mathbf{i}}$, with the same proviso about the similarity type) by the equations E3 displayed above. In particular, when 0 is included in the similarity type:

- The already mentioned variety of *MV-algebras* is term equivalent to the subvariety $\mathcal{MV}$ of $\mathcal{IGMV}$ whose equational basis relative to $\mathcal{IGMV}$ is given by the equations E1 and

  E5 $\quad xy \approx yx.$

- The variety of *pseudo-MV-algebras*, a noncommutative generalization of MV algebras [12], is therefore term equivalent to the subvariety $\mathcal{P}s\mathcal{MV}$ of $\mathcal{IGMV}$ whose equational basis relative to $\mathcal{IGMV}$ is given just by the equation E1.
- It easily follows from our preceding remarks that the variety of *Boolean algebras* is term equivalent to the subvariety $\mathcal{BA}$ of $\mathcal{IGMV}$ whose equational basis relative to $\mathcal{IGMV}$ is given by the equations E1 and E2.

Finally, given an FL $\mathbf{A} = (A, \vee, \wedge, \cdot, \backslash, /, 1, 0)$, an element $a \in A$ is said to be *dualizing* if $0/(a\backslash 0) = a = (0/a)\backslash 0$, and $\mathbf{A}$ itself is said to be *involutive* if every member of A is dualizing. We denote by $\mathcal{InFL}$ the variety of all involutive FLs. $\mathcal{PsMV}$, and consequently also $\mathcal{MV}$ and $\mathcal{BA}$, are all subvarieties of $\mathcal{InFL}$.

We remark, for future reference, that $\mathcal{RL}$ is both congruence permutable (witness the term $[z \vee (z/y)x] \wedge [x \vee (x/y)z]$) and 1-regular (witness the terms $x \backslash y \wedge 1, y \backslash x \wedge 1$). Since Mal'cev properties carry over to expansions, $\mathcal{FL}$ has these properties as well. Any variety which is congruence permutable and 1-regular is, in particular, *ideal determined*: the lattice of congruence relations and the lattice of ideals (in the sense of [16]) of any algebra in the variety are isomorphic. It is proved in [3] (see also [10]) that for any RL or FL $\mathbf{A}$, ideals of $\mathbf{A}$ coincide with convex normal subalgebras of $\mathbf{A}$.

4. Joins and products of residuated lattice varieties

Recall the following definitions from universal algebra:

Definition 8. ([14]) Two varieties $\mathcal{U}$ and $\mathcal{V}$, of the same similarity type ν, are said to be *independent* if there exists a binary term $*$ such that

$$\mathcal{U} \vDash x * y \approx x, \quad \mathcal{V} \vDash x * y \approx y.$$

Also, $\mathcal{U}$ and $\mathcal{V}$ are said to be *disjoint* if their intersection is the trivial variety of type ν.

Independent varieties are always disjoint, but the converse is not necessarily true (for a counterexample, see e.g. [23]). In [14] it is shown that if $\mathcal{U}$ and $\mathcal{V}$ are independent varieties, then they are disjoint and their join $\mathcal{U} \vee \mathcal{V}$ in the lattice of all varieties of type ν coincides with their *direct product*, i.e. with the class

$$\mathcal{U} \times \mathcal{V} = \{\mathbf{A} \times \mathbf{B} : \mathbf{A} \in \mathcal{U}, \mathbf{B} \in \mathcal{V}\}.$$

Once again the converse need not be true, but it is proved in [19] that if $\mathcal{U}$ and $\mathcal{V}$ are disjoint subvarieties of a *congruence permutable variety*[2], then they are independent and $\mathcal{U} \vee \mathcal{V} = \mathcal{U} \times \mathcal{V}$.

This result is relevant for our problem in that the varieties $\mathcal{RL}$ and $\mathcal{FL}$ are, as recalled above, congruence permutable. Moreover, $\mathcal{IRL}$ and $\mathcal{LG}$

[2] Actually, for a pointed variety $\mathcal{V}$ the assumption of congruence permutability is unnecessarily strong: 1-permutability, i.e. the property that for every $\mathbf{A} \in \mathcal{V}$ and every $\theta, \psi \in \mathrm{Con}(\mathbf{A})$ $1/\theta \circ \psi = 1/\psi \circ \theta$, together with the presence of a unital groupoid term reduct, suffices [21].

are clearly disjoint subvarieties of $\mathcal{RL}$; they are also independent, witness the term

$$x * y = x\,(x\backslash 1)\,((y\backslash 1)\backslash 1)\,.$$

The same relation holds true for $\mathcal{FL}_{\mathbf{i}}$ and $\mathcal{LG}$ as subvarieties of $\mathcal{FL}$, and all the more so if we replace $\mathcal{FL}_{\mathbf{i}}$ by any of its proper subvarieties. In particular, $\mathcal{BA}$ and $\mathcal{LG}$ are disjoint and independent subvarieties of the congruence permutable variety $\mathcal{FL}$, whence we are entitled to assume that $\mathcal{BA} \vee \mathcal{LG} = \mathcal{BA} \times \mathcal{LG}$. The problem of axiomatizing their join, therefore, reduces to the more accessible problem of axiomatizing their direct product.

The independence of $\mathcal{IRL}$ ($\mathcal{FL}_{\mathbf{i}}$) and $\mathcal{LG}$ implies that the integral elements of an RL (FL) are precisely those elements of the form $a\,(a\backslash 1)$, and that the invertible elements are precisely those elements of the form $(a\backslash 1)\backslash 1$. In what follows, if t is an RL or FL term, we abbreviate by $\widetilde{t}$ the term $t\,(t\backslash 1)$.

In [19], Jónsson and one of the present authors provide necessary and sufficient conditions for an RL to be represented as the direct product of an integral RL and an ℓ-group. The following theorem, in fact, is proved:

Theorem 9. *For an RL* $\mathbf{A}$*, the following conditions are equivalent:*

(1) *There exist an integral RL* $\mathbf{B}$ *and an* ℓ*-group* $\mathbf{C}$ *such that* $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$*;*

(2) $\mathbf{A}$ *satisfies*

E6 $\quad (x\backslash 1 \vee y)\,((x\backslash 1 \vee y)\backslash 1) \approx 1$

Proof. (sketch).
(1) clearly implies (2), for $(x\backslash 1 \vee y)\,((x\backslash 1 \vee y)\backslash 1)^{\mathbf{B}\times\mathbf{C}}\,((b_1, c_1), (b_2, c_2))$ simplifies to

$$\begin{aligned} \left((1, c_1^{-1}) \vee (b_2, c_2)\right)\left((1, c_1^{-1})\backslash(1,1) \wedge (b_2, c_2)\backslash(1,1)\right) &= \\ \left((1, c_1^{-1}) \vee (b_2, c_2)\right)\left((1, c_1) \wedge \left(1, c_2^{-1}\right)\right) &= \\ \left(1, c_1^{-1} \vee c_2\right)\left(1, c_1 \wedge c_2^{-1}\right) &= (1,1)\,. \end{aligned}$$

As to the converse implication, let $\mathbf{A}$ satisfy E6, and let B and C be, respectively, the set of all integral elements and the set of all invertible elements of A. B and C are subuniverses of $\mathbf{A}$, and the corresponding subalgebras $\mathbf{B}$ and $\mathbf{C}$ are, respectively, an integral RL and an ℓ-group. Now, for any $a \in A$ let

$$f(a) = (a\,(a\backslash 1)\,, (a\backslash 1)\backslash 1)\,.$$

f provides the required isomorphism between $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$:

- it is well-defined, for all elements of the form $a\,(a\backslash 1)$ are integral and all elements of the form $(a\backslash 1)\backslash 1$ are invertible;
- it is onto, for all integral elements have the form $a\,(a\backslash 1)$ and all invertible elements have the form $(a\backslash 1)\backslash 1$;
- it is one-one, for it can be proved that if an RL $\mathbf{A}$ satisfies E6, then for any $a \in A$
$$a = a\,(a\backslash 1)\,((a\backslash 1)\backslash 1)\,;$$
- it preserves all the residuated lattice operations, by properties of RLs.

■

In particular, when $\mathbf{B}$ in the preceding theorem is a GMV-algebra, we obtain the following proposition, first proved using different methods in [11]:

THEOREM 10. *An RL* $\mathbf{A}$ *is a GMV-algebra iff there exist an integral GMV-algebra* $\mathbf{B}$ *and an* ℓ*-group* $\mathbf{C}$ *such that* $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$.

5. An equational basis for $\mathcal{BA} \vee \mathcal{LG}$

Theorem 10 almost caters for all our needs, since it gives necessary and sufficient conditions for an RL to be representable as a direct product of an integral GMV-algebra and an ℓ-group. As we have seen, however, if we want to conveniently express Boolean algebras the type of residuated lattices is not enough: we need the additional constant 0 which realizes the bottom element in any Boolean algebra and the identity 1 in any ℓ-group. The appropriate setting for our common generalization, therefore, is the variety $\mathcal{FL}$ of FL-algebras. Should Theorem 10 carry over to FL-algebras - that is, should it be true that an FL $\mathbf{A}$ is the product of an integral GMV-algebra (or a pseudo-MV algebra) and of an ℓ-group if and only if it is a GMV-algebra, possibly satisfying additional conditions about the newly added constant - then we would be done because, as we have seen, an equational basis for $\mathcal{BA}$ relative to $\mathcal{GMV}$ is known. It would suffice, then, to add axioms expressing the condition that the *integral* elements in $\mathbf{A}$ satisfy the equations in such a basis. Our first goal, then, is generalizing Theorem 10 to the new setting. We will do this for the more general case of Theorem 9.

Given an FL $\mathbf{A}$, we denote hereafter by $\mathbf{A}^*$ its RL (i.e. its 0-free) reduct. Consider the following equations:

E7 $0^2 \approx 0$

E8 $0\backslash\widetilde{x} \approx 1$

It is possible to show that:

LEMMA 11. *If an FL* $\mathbf{A}$ *satisfies E6 and E8, then it also satisfies E7.*

PROOF. By Theorem 9, if $\mathbf{A}$ satisfies E6 there exist $\mathbf{B}^* \in \mathcal{IRL}$ and $\mathbf{C} \in \mathcal{LG}$ such that $\mathbf{A}^* \simeq \mathbf{B}^* \times \mathbf{C}$; more precisely, $\mathbf{B}^*$ is the subalgebra of integral elements of $\mathbf{A}^*$, and $\mathbf{C}$ is the subalgebra of invertible elements of $\mathbf{A}^*$. By our observation on the form of integral elements in FL algebras, it follows from E8 that 0 is less or equal than any element of $\mathbf{B}^*$; if we can show that 0 is *itself* integral, we are done, since the bottom element of any residuated lattice (and thus, in particular, of $\mathbf{B}^*$) is necessarily idempotent. However, assigning the value 1 to the variable x in E8, we obtain that $0\backslash 1 = 1$, whence $0 = 0 \cdot 1 = 0\,(0\backslash 1)$ has the required form, proving our claim. ∎

LEMMA 12. *An FL* $\mathbf{A}$ *belongs to* $\mathcal{FL}_{\mathbf{w}} \times \mathcal{LG}$ *if and only if it satisfies E6 and E8.*

PROOF. E6 and E8 hold in both $\mathcal{FL}_{\mathbf{w}}$ and $\mathcal{LG}$, hence in $\mathcal{FL}_{\mathbf{w}} \times \mathcal{LG}$. Conversely, let $\mathbf{A}$ be an FL that satisfies E6 and E8. In view of Lemma 11, it also satisfies E7. As already observed, moreover, Theorem 9 implies that there exist $\mathbf{B}^* \in \mathcal{IRL}$ and $\mathbf{C} \in \mathcal{LG}$ such that $\mathbf{A}^* \simeq \mathbf{B}^* \times \mathbf{C}$, where $\mathbf{B}^*$ is the subalgebra of integral elements of $\mathbf{A}^*$, $\mathbf{C}$ is the subalgebra of invertible elements of $\mathbf{A}^*$, and the map $h\,(b, c) = bc$ is an isomorphism from $\mathbf{B}^* \times \mathbf{C}$ onto $\mathbf{A}^*$. Note next that, in light of E8, $0^{\mathbf{A}}$ is the least element of $\mathbf{B}^*$. Thus the linguistic expansion $\mathbf{B}$ of $\mathbf{B}^*$ by the constant 0 is a member of $\mathcal{FL}_{\mathbf{w}}$. It follows that the expansion $\mathbf{B} \times \mathbf{C}$ of $\mathbf{B}^* \times \mathbf{C}$ by the constant 0, where $0^{\mathbf{B}\times\mathbf{C}} = (0^{\mathbf{B}}, 1^{\mathbf{C}})$, satisfies E8. Lastly, the equalities $h(0, 1) = (0).^{\mathbf{A}}(1) = 0$, show that h is an isomorphism between the FLs $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$. ∎

COROLLARY 13. *An FL* $\mathbf{A}$ *belongs to* $\mathcal{P}s\mathcal{MV} \times \mathcal{LG}$ *if and only if it is a GMV-algebra satisfying E8.*

PROOF. From Theorem 10 and Lemma 12. ∎

From now on, by "GMV-algebra" we will invariably mean an algebra of the same type as FL algebras, resolving once and for all the ambiguity concerning the presence of 0 in the type.

THEOREM 14. *An FL* $\mathbf{A}$ *belongs to* $\mathcal{BA} \times \mathcal{LG}$ *if and only if it is a GMV-algebra satisfying E8 and*

E9 $\widetilde{x}\widetilde{y} \approx \widetilde{x} \wedge \widetilde{y}$

Proof.

Left to right: If $\mathbf{A} \simeq \mathbf{B} \times \mathbf{C}$ belongs to $\mathcal{BA} \times \mathcal{LG}$, then a fortiori it belongs to $\mathcal{P}s\mathcal{MV} \times \mathcal{LG}$, whence by Corollary 13 it is a GMV-algebra satisfying E8. Now, recall that $\mathcal{BA}$ is axiomatized relative to $\mathcal{P}s\mathcal{MV}$ by E2. If the RL subalgebra $\mathbf{B}$ of integral elements of $\mathbf{A}$ is Boolean, then it satisfies such an equation, that is to say $\mathbf{A}$ satisfies E9.

Right to left: If $\mathbf{A}$ is a GMV-algebra satisfying E8, by Corollary 13 the integral elements of $\mathbf{A}$ are the universe of an algebra $\mathbf{B} \in \mathcal{P}s\mathcal{MV}$, while the invertible elements of $\mathbf{A}$ are the universe of an algebra $\mathbf{C} \in \mathcal{LG}$, and $\mathbf{A} \simeq \mathbf{B} \times \mathbf{C}$. However, since $\mathbf{A}$ satisfies E9, $\mathbf{B} \in \mathcal{BA}$, whence our conclusion. ■

Let $\mathcal{V}$ be the variety of GMV algebras whose equational basis relative to $\mathcal{GMV}$ is given by E8 and

E10 $\quad x\,(x\backslash 1 \wedge y\backslash 1)\,y \approx\ x \wedge y.$

Eventually, we want to show that $\mathcal{V}$ coincides with $\mathcal{BA} \times \mathcal{LG}$ - and thus with $\mathcal{BA} \vee \mathcal{LG}$, in virtue of our remarks in Section 4. However, we start by establishing a few arithmetical properties of $\mathcal{V}$ that may be of independent interest.

Lemma 15. *$\mathcal{V}$ satisfies the following equations and quasiequations:*

(i) $(x\backslash 1 \vee y)\,((x\backslash 1 \vee y)\,\backslash 1) \approx 1$	(iii) $x\,(x\backslash 1)\,x \approx x$;
(ii) $(x\backslash 1)\,((x\backslash 1)\,\backslash 1) \approx 1$;	(iv) $0 \leq x \Rightarrow x \leq x^2$;

Proof.

- (i) This equation holds in all GMV-algebras, as proved in [19].
- (ii) Replace y by $x\backslash 1$ in (i).
- (iii) Replace y by x in E10.
- (iv) Suppose $0 \leq a$. By the proof of Lemma 11 and Proposition 6, $a\backslash 1 \leq 0\backslash 1 = 1$. Then by (iii) and Proposition 6,

$$a = a\,(a\backslash 1)\,a \leq a1a = a^2.$$ ■

We are now ready to prove our main theorem:

Theorem 16. *$\mathcal{V} = \mathcal{BA} \times \mathcal{LG} = \mathcal{BA} \vee \mathcal{LG}$. In other words, $\mathcal{BA} \vee \mathcal{LG}$ is axiomatized relative to $\mathcal{FL}$ by*

E4 $\quad x/\,((x \vee y)\,\backslash x) \approx x \vee y \approx (x/\,(x \vee y))\,\backslash x;$

E8 $0\backslash\left(x\left(x\backslash 1\right)\right)\approx 1$*; and*

E10 $x\left(x\backslash 1\wedge y\backslash 1\right)y\approx\ x\wedge y$.

Proof. The equation E10 holds in both $\mathcal{BA}$ and $\mathcal{LG}$, hence in $\mathcal{BA}\times\mathcal{LG}$. Conversely, it suffices to show that E9 holds in $\mathcal{V}$. Let $\mathbf{A}\in\mathcal{V}$ and let $a,b\in\mathbf{A}$. Then:

$$\begin{aligned}\widetilde{a}\wedge\widetilde{b}\ &=\widetilde{a}\left(\widetilde{a}\backslash 1\wedge\widetilde{b}\backslash 1\right)\widetilde{b} && \text{(E10)}\\ &=\widetilde{a}\left(1\wedge 1\right)\widetilde{b} && \text{(Proposition 6)}\\ &=\widetilde{a}\widetilde{b} && \text{(res. lattice axioms)}\end{aligned}$$

■

Proposition 17.

(1) *$\mathcal{MV}\times\mathcal{LG}=\mathcal{MV}\vee\mathcal{LG}$ is axiomatized relative to $\mathcal{FL}$ by the equations*

E4 $x/\left(\left(x\vee y\right)\backslash x\right)\approx x\vee y\approx\left(x/\left(x\vee y\right)\right)\backslash x$;

E8 $0\backslash\left(x\left(x\backslash 1\right)\right)\approx 1$*; and*

E11 $x\left(x\backslash 1\wedge y\backslash 1\right)y\approx y\left(x\backslash 1\wedge y\backslash 1\right)x$.

(2) *$\mathcal{PsMV}\times\mathcal{LG}=\mathcal{PsMV}\vee\mathcal{LG}$ is axiomatized relative to $\mathcal{FL}$ by the equations*

E4 $x/\left(\left(x\vee y\right)\backslash x\right)\approx x\vee y\approx\left(x/\left(x\vee y\right)\right)\backslash x$; *and*

E8 $0\backslash\left(x\left(x\backslash 1\right)\right)\approx 1$.

Proof. Given $\mathbf{A}\simeq\mathbf{B}\times\mathbf{C}$ in $\mathcal{MV}\vee\mathcal{LG}$, E11 guarantees that the set of integral elements in the decomposition provided by Lemma 12 satisfies $xy\approx yx$. However, we already observed that such an equation axiomatizes $\mathcal{MV}$ relative to $\mathcal{IGMV}$. Without that condition, we only know that the set of integral elements is the universe of a lower bounded integral GMV algebra with bottom element 0 - i.e. a pseudo-MV algebra (cp. Corollary 13). ■

Proposition 18. *The axioms of $\mathcal{BA}\vee\mathcal{LG}$ are independent (relative to $\mathcal{FL}$).*

Proof. Each of the next three examples of FLs satisfies all the axioms of $\mathcal{BA}\vee\mathcal{LG}$ except for the indicated one.

(E4) Consider any direct product of a Heyting algebra and an ℓ-group.

(E8) Let $\mathbf{A}$ be the 4-element Boolean algebra, and let $0^{\mathbf{A}}$ be one of the atoms of the algebra.

(E10) It suffices to consider the 3-element MV-chain. ■

6. Common metatheoretical properties

In this section we strengthen our case in favour of $\mathcal{BA} \vee \mathcal{LG}$ as a suitable common abstraction of Boolean algebras and ℓ-groups by showing that it shares with $\mathcal{BA}$ and $\mathcal{LG}$ a number of the nice universal algebraic properties characterizing them. In particular:

(1) $\mathcal{BA} \vee \mathcal{LG}$ is a congruence regular, 1-ideal determined and arithmetical variety.

(2) Members of this variety have representations in terms of residuated endomorphisms of chains — a property which fails to hold for generic FLs. This result generalizes both Holland's well-known representation theorem for ℓ-groups in terms of algebras of order-preserving automorphisms of chains (see e.g. [7]), the analogous representation of Boolean algebras as subdirect products of two-element chains, and the representation of commutative ℓ-monoids (definition is given below) in [24].

As to the first item in our list above, there is not much to say. It is well-known that $\mathcal{FL}$ is in itself both ideal determined and arithmetical (i.e. congruence permutable and congruence distributive), and such properties automatically transfer to subvarieties. That $\mathcal{BA} \vee \mathcal{LG}$ is *congruence regular* - meaning that, given an arbitrary $\mathbf{A} \in \mathcal{BA} \vee \mathcal{LG}$, no two different congruences of $\mathbf{A}$ share a coset - is just slightly less obvious. It is known, in fact, that congruence regularity is a Mal'cev property [13]: a variety $\mathcal{V}$ is congruence regular iff there are ternary terms $t_1, ..., t_n$ such that every algebra in the variety satisfies the quasiequations

$$x \approx y \Longleftrightarrow t_1(x, y, z) \approx z \& \ldots \& t_n(x, y, z) \approx z.$$

However, although $\mathcal{BA}$ and $\mathcal{LG}$ are both congruence regular, $\mathcal{FL}$ is not. In addition, congruence regularity is witnessed in $\mathcal{BA}$ and $\mathcal{LG}$ by terms that look very much different from each other: $x(y\backslash 1)z$ for ℓ-groups, $(x \wedge y\backslash 0) \vee (y \wedge x\backslash 0) \vee z$ for Boolean algebras. Nonetheless, a result in [14] implies that every Mal'cev property transfers to joins of independent varieties, provided only that both joinands have it.

As to the second item, recall that an *ℓ-monoid* is an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, 1)$ of type $(2, 2, 2, 0)$ such that

- $(L, \wedge, \vee)$ is a lattice;
- $(L, \cdot, 1)$ is a monoid; and

- for all $a, b, c, d \in L$,

$$\begin{aligned} a\,(b \vee c)\,d &= abd \vee acd \\ a\,(b \wedge c)\,d &= abd \wedge acd \end{aligned}$$

Homomorphisms of ℓ-monoids are referred to as *ℓ-homomorphisms*, and we feel free once again to use plain juxtaposition in place of $\cdot$ in what follows.

We introduce a relation between elements of an ℓ-monoid which will turn out to be extremely useful throughout this section. If $\mathbf{L}$ is an ℓ-monoid and $\emptyset \neq H \subseteq L$, we define:

$$\begin{aligned} H/a &= \{c \in L : ca \in H\} \\ a \sim_H b &\Leftrightarrow H/a = H/b \\ &\Leftrightarrow \forall c \in L\,(ca \in H \Leftrightarrow cb \in H) \end{aligned}$$

It is clear that $\sim_H$ is an equivalence relation.

Now, given any chain $\mathbf{\Omega}$, the set $\mathcal{E}nd(\mathbf{\Omega})$ of all order-endomorphisms of $\mathbf{\Omega}$ (i.e., of all order-preserving maps on $\mathbf{\Omega}$) is (the universe of) a monoid with respect to function composition, and a lattice with respect to pointwise join and meet; moreover, we have for all such endomorphisms f, g, h, k:

$$\begin{aligned} f\,(g \vee h)\,k &= fgk \vee fhk \\ f\,(g \wedge h)\,k &= fgk \wedge fhk \end{aligned}$$

It follows that $\mathcal{E}nd\,(\mathbf{\Omega})$ is the universe of an ℓ-monoid whose lattice reduct is *distributive*. By abuse of notation, we denote such an ℓ-monoid by the same label $\mathcal{E}nd\,(\mathbf{\Omega})$. Also $\mathcal{A}ut\,(\mathbf{\Omega})$, the set of all *automorphisms* of $\mathbf{\Omega}$, is the universe of an ℓ-monoid which is actually an ℓ-group. Holland's Theorem [17], which we generalize in Theorem 25 below, states that also the converse is true — that is, any ℓ-group can be embedded into $\mathcal{A}ut\,(\mathbf{\Omega})$ for some chain $\mathbf{\Omega}$.

The first two lemmas below, whose proofs are presented below for the reader's convenience, are due to Anderson and Edwards [1]. The key idea of Lemma 19 is due to Merlier [24].

Lemma 19. *For an ℓ-monoid $\mathbf{L}$, the following statements are equivalent:*

(1) *There exists a non-trivial ℓ-homomorphism $\varphi : \mathbf{L} \to \mathcal{E}nd(\mathbf{\Omega})$, for some chain $\mathbf{\Omega}$.*

(2) *$\mathbf{L}$ has a (proper) prime lattice ideal.*

Proof. (1) $\Rightarrow$ (2). Let $\varphi : \mathbf{L} \to \mathcal{E}nd\,(\mathbf{\Omega})$ be a nontrivial ℓ-homomorphism. Then there exist elements $a, b \in L$ such that $\varphi(a) \neq \varphi(b)$. Since φ preserves lattice operations, we may assume that $a < b$ and $\varphi(a) < \varphi(b)$. Hence, there exists $\omega \in \mathbf{\Omega}$ such that $\varphi(a)(\omega) < \varphi(b)(\omega)$. Fix such an ω, and let

$$H_\omega = \{c \in L : \varphi(c)(\omega) \leq \varphi(a)(\omega)\}.$$

H_ω is both nonempty, as $a \in H_\omega$, and proper, as $b \notin H_\omega$. It is also easy to check that it is a prime lattice ideal of $\mathbf{L}$.

(2) $\Rightarrow$ (1). Let H be a prime ideal of $\mathbf{L}$. We first show that $\{H/a : a \in L\}$ is a chain under set inclusion. Indeed, suppose that $H/a \subsetneq H/b$. Then there exists $c \in L$ such that $ca \in H$ and $cb \notin H$. Now $c(a \wedge b) \leq ca \in H$, whence $c(a \wedge b) \in H$. Also, for all $d \in H/b$, $db \in H$, whence $d(a \wedge b) \in H$. It follows that

$$(c \vee d)\,a \wedge (c \vee d)\,b = (c \vee d)(a \wedge b) = c(a \wedge b) \vee d(a \wedge b) \in H.$$

Note that $(c \vee d)\,b \notin H$, since $cb \leq (c \vee d)\,b$ and $cb \notin H$. However, H is a prime ideal and thus we must have that $(c \vee d)\,a \in H$, whence a fortiori $da \in H$, i.e. $d \in H/a$. We have shown that $H/b \subseteq H/a$, that is, $\{H/a : a \in L\}$ is a chain under set inclusion.

Next, we prove that $\sim_H$is a lattice congruence compatible with left multiplication. We will confine ourselves to showing that for all $a, b, c, d \in L$: (i) if $a \sim_H b$, then $ca \sim_H cb$; (ii) if $a \sim_H a'$ and $b \sim_H b'$, then $a \wedge b \sim_H a' \wedge b'$. As to (i), $a \sim_H b$ means by definition that $H/a = H/b$, whence

$$H/ca = (H/a)/c = (H/b)/c = H/cb,$$

i.e. $ca \sim_H cb$. As to (ii), suppose that $a \sim_H a'$ and $b \sim_H b'$, and let $c \in L$. Since H is a prime ideal, $ca \wedge cb = c(a \wedge b) \in H$ iff $ca \in H$ or $cb \in H$. By hypothesis, this happens iff $ca' \in H$ or $cb' \in H$, i.e. iff $c(a' \wedge b') = ca' \wedge cb'$ $\in H$. Thus $a \wedge b \sim_H a' \wedge b'$.

Now, let $\mathbf{\Omega}_H$ denote the quotient $\mathbf{L}^*/\sim_H$, where $\mathbf{L}^*$ is the lattice reduct of $\mathbf{L}$. To verify that $\mathbf{\Omega}_H$ is a chain, observe that

$$\begin{aligned}
a/\sim_H \leq b/\sim_H \quad & \text{iff } a \vee b/\sim_H = b/\sim_H \\
& \text{iff } H/a \vee b = H/b \\
& \text{iff } \forall c\,(c(a \vee b) \in H \Leftrightarrow cb \in H) \\
& \text{iff } \forall c\,(ca \vee cb \in H \Leftrightarrow cb \in H) \\
& \text{iff } \forall c\,(cb \in H \Rightarrow ca \in H) \\
& \text{iff } H/b \subseteq H/a.
\end{aligned}$$

Thus $\mathbf{\Omega}_H$ is a chain, since $\{H/a : a \in L\}$ is such.

Lastly, define $\varphi : \mathbf{L} \to \mathcal{E}nd\,(\mathbf{\Omega}_H)$ by

$$\varphi\,(a)\,(b/\sim_H) = ab/\sim_H .$$

Note that for a given $a \in L$, $\varphi\,(a)$ is an order-endomorphism of $\mathbf{\Omega}_H$: if $b/\sim_H \leq c/\sim_H$, then $H/c \subseteq H/b$ and so, for all $d \in L$, $dab \in H$ whenever $dac \in H$, i.e., $H/ac \subseteq H/ab$ and $ab/\sim_H \leq ac/\sim_H$. Lastly, it is easy to verify that φ is an ℓ-homomorphism of $\mathbf{L}$ into $\mathcal{E}nd\,(\mathbf{\Omega}_H)$. ■

We can now strengthen the result of Lemma 19 as follows:

Lemma 20. *For an ℓ-monoid $\mathbf{L}$, the following statements are equivalent:*

(1) $\mathbf{L}$ *is distributive.*

(2) *There exists a chain $\mathbf{\Omega}$ such that $\mathbf{L}$ can be embedded into $\mathcal{E}nd\,(\mathbf{\Omega})$.*

Proof. We only prove (1) $\to$ (2), since the other implication is immediate. Suppose that $\mathbf{L}$ is distributive. We first show that there exists a family of chains $\{\mathbf{\Omega}_i\}_{i\in I}$ such that $\mathbf{L}$ can be embedded into $\prod_{i\in I}(\mathcal{E}nd\,(\mathbf{\Omega}_i))$. Let $\{H_i\}_{i\in I}$ be the collection of all prime ideals of $\mathbf{L}$ and let $\mathbf{\Omega}_{H_i}$ (henceforth abbreviated as $\mathbf{\Omega}_i$) be constructed from H_i as in the preceding lemma. Also, let $\varphi_i : \mathbf{L} \to \mathcal{E}nd\,(\mathbf{\Omega}_i)$ be the corresponding homomorphism introduced in the proof of the same lemma. Note that $\varphi_i\,(a) = \varphi_i\,(b)$ implies that for all c, $ac/\sim_{H_i} = bc/\sim_{H_i}$, whence $a/\sim_{H_i} = b/\sim_{H_i}$ and so $a \sim_{H_i} b$. Therefore, in order to show that $\mathbf{L}$ can be embedded into $\prod_{i\in I}(\mathcal{E}nd\,(\mathbf{\Omega}_i))$, it will suffice to prove that $\bigcap_{i\in I}\{\sim_{H_i}\} = \Delta_{\mathbf{L}}$, the identity congruence of $\mathbf{L}$. To this end, let $a \neq b$ in L. The prime ideal separation theorem for distributive lattices guarantees that there exists a prime ideal H_i such that (without loss of generality) $a \in H_i, b \notin H_i$. This, however, means that $a \nsim_{H_i} b$, showing that $\bigcap_{i\in I}\{\sim_{H_i}\} = \Delta_{\mathbf{L}}$.

Let now $\{\mathbf{\Omega}_i\}_{i\in I}$ be a family of chains such that $\mathbf{L}$ can be embedded into $\prod_{i\in I}(\mathcal{E}nd\,(\mathbf{\Omega}_i))$. Let $\preceq$ be a total order on I and let $\mathbf{\Omega} = \bigcup_{i\in I}\{\mathbf{\Omega}_i\}$. We may assume that the chains $\mathbf{\Omega}_i$ are pairwise disjoint. Provide $\mathbf{\Omega}$ with the following lexicographical order $\leq^{\mathbf{\Omega}}$: for $a, b \in \mathbf{\Omega}$,

$$a \leq^{\mathbf{\Omega}} b \text{ iff } \begin{cases} i \prec j,\ a \in \mathbf{\Omega}_i \text{ and } b \in \mathbf{\Omega}_j; \text{ or} \\ i = j \text{ and } a \leq^{\mathbf{\Omega}_i} b. \end{cases}$$

Finally, define $\Phi : \prod_{i\in I}(\mathcal{E}nd\,(\mathbf{\Omega}_i)) \to \mathcal{E}nd\,(\mathbf{\Omega})$ by $\Phi(f)(a) = f(i)(a)$, for all $a \in \Omega_i$ and $i \in I$. A routine check shows that Φ is an ℓ-embedding. ■

Any FL algebra $\mathbf{A}$ in $\mathcal{BA} \vee \mathcal{LG}$ has an ℓ-monoid reduct, for multiplication distributes from both sides over both meet and join. Such a reduct is distributive, whence Lemma 20 applies and we have a representation in terms of order-endomorphisms of a chain Ω. We show below that a careful choice of Ω leads to a representation that involves only the residuated maps on it.

Let $\mathbf{P}$ and $\mathbf{Q}$ be posets. Recall that a map $f : \mathbf{P} \to \mathbf{Q}$ is *residuated* if there exists a map $f^* : \mathbf{Q} \to \mathbf{P}$ such that for any $a \in P$ and any $b \in Q$, $f(a) \leq^{\mathbf{Q}} b$ iff $a \leq^{\mathbf{P}} f^*(b)$. In this case, we say that f and f^* form a *residuated pair*, and that f^* is a *residual* of f. Note that a binary map is residuated in the sense of Section 3 if and only if all translates of the map are residuated in the preceding sense. We take a note without a proof of the following well-known result:

Lemma 21. *If $\mathbf{P}$ and $\mathbf{Q}$ are complete lattices, then $f : \mathbf{P} \to \mathbf{Q}$ is residuated iff f preserves arbitrary joins.*

Given a chain $\mathbf{\Omega}$, we ambiguously denote by $\mathcal{R}es\,(\mathbf{\Omega})$ both the set of all residuated maps on $\mathbf{\Omega}$ and the ℓ-monoid which has the same set as a universe. We have the following:

Lemma 22. *If $\mathbf{\Omega}$ is a chain, $\mathcal{R}es\,(\mathbf{\Omega})$ is an ℓ-submonoid of $\mathcal{E}nd\,(\mathbf{\Omega})$: In particular, for any $f, g \in \mathcal{R}es\,(\mathbf{\Omega})$: (i) $(fg)^* = g^* f^*$; (ii) $(f \vee g)^* = f^* \wedge g^*$; (iii) $(f \wedge g)^* = f^* \vee g^*$.*

Proof. We verify (iii) as a means of an example. We have that

$$\begin{aligned} a \leq (f^* \vee g^*)(b) \quad & \text{iff } a \leq f^*(b) \text{ or } a \leq g^*(b) \\ & \text{iff } f(a) \leq b \text{ or } g(a) \leq b \\ & \text{iff } f(a) \wedge g(a) \leq b \\ & \text{iff } (f \wedge g)(a) \leq b, \end{aligned}$$

where the first equivalence holds true because $\mathbf{\Omega}$ is a chain. ∎

A subset I of a poset $\mathbf{P}$ is said to be a *lower set* of $\mathbf{P}$ if whenever $y \in P$, $x \in I$, and $y \leq x$, then $y \in I$. Note that the empty set $\emptyset$ is a lower set. A *principal* lower set is a lower set of the form $\downarrow a = \{x \in P \mid x \leq a\}$. More generally, for $A \subseteq P$, $\downarrow A = \{x \in P \mid x \leq a, \text{ for some } a \in A\}$ denotes the smallest lower set containing A. The set $\mathcal{L}(\mathbf{P})$ of lower sets of $\mathbf{P}$ ordered by set inclusion is a complete lattice; the join is the set-union, and the meet is the set-intersection. We remark that the map $a \mapsto \downarrow a$ $(a \in P)$ is an isomorphism between $\mathbf{P}$ and the subposet $\dot{\mathbf{P}} = \{\downarrow a \mid a \in P\}$ of $\mathcal{L}(\mathbf{P})$.

Lemma 23. *Every order-preserving map f on a poset $\mathbf{P}$ induces a residuated map $\widehat{f}$ on $\mathcal{L}(\mathbf{P})$. Moreover, the correspondence $f \mapsto \widehat{f}$ is bijective.*

Proof. In view of the discussion in the preceding paragraph, we may assume that f is an order preserving map on $\dot{\mathbf{P}}$. Given $I \in \mathcal{L}(\mathbf{P})$ and a representation $I = \bigcup_{a\in A} \downarrow a$, $(A \subseteq P)$ of I as a union of principal lower sets, we define $\widehat{f}(I) = \bigcup_{a\in A} f(\downarrow a)$. We first observe that $\widehat{f}(I)$ is well-defined, that is, it does not depend on the representation of I. Indeed suppose that $I = \bigcup_{a\in A} \downarrow a = \bigcup_{b\in B} \downarrow b$, for $A, B \subseteq P$. We need to verify that $\bigcup_{a\in A} f(\downarrow a) = \bigcup_{b\in B} f(\downarrow b)$. Let $a \in A$. We have $\downarrow a \subseteq \bigcup_{b\in B} \downarrow b$, and consequently there exists $b_0 \in B$ such that $\downarrow a \subseteq \downarrow b_0$. Since f is an order-preserving map on $\dot{\mathbf{P}}$, we have the inclusions $f(\downarrow a) \subseteq f(\downarrow b_0) \subseteq \bigcup_{b\in B} f(\downarrow b)$. Thus, $\bigcup_{a\in A} f(\downarrow a) \subseteq \bigcup_{b\in B} f(\downarrow b)$, and the reverse inclusion follows by symmetry. That $\widehat{f}$ is a residuated map follows directly from Lemma 21. It is also clear that the correspondence $f \mapsto \widehat{f}$ is bijective, since the restriction of $\widehat{f}$ to $\dot{\mathbf{P}}$ is f. ■

We remark that if $\mathbf{\Omega}$ is a chain, then $\mathcal{L}(\mathbf{\Omega})$ is a complete chain.

Lemma 24. *If $\mathbf{\Omega}$ is a chain, then the ℓ-monoid $\mathcal{E}nd(\mathbf{\Omega})$ can be embedded into the ℓ-monoid $\mathcal{R}es(\mathcal{L}(\mathbf{\Omega}))$.*

Proof. We leave to the reader the easy task of verifying that the map $f \mapsto \widehat{f}$ of the preceding result provides the required embedding. ■

Combining Lemma 20 and Lemma 24, we obtain the following generalization of Holland's Representation Theorem:

Theorem 25. *For an ℓ-monoid $\mathbf{L}$, the following statements are equivalent:*

(1) $\mathbf{L}$ *is distributive.*

(2) *There exists a chain $\mathbf{\Omega}$ such that $\mathbf{L}$ can be embedded as an ℓ-monoid into $\mathcal{R}es(\mathbf{\Omega})$.*

In particular we have:

Corollary 26. *An RL $\mathbf{A}$ can be embedded as an ℓ-monoid into $\mathcal{R}es(\mathbf{\Omega})$, for some chain $\mathbf{\Omega}$, iff it satisfies the equations*

E12 $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge w)$; *and*

E13 $x(y \wedge z)w \approx xyw \wedge xzw$.

Corollary 27. *If* $\mathbf{A} \in \mathcal{BA} \vee \mathcal{LG}$*, then there exists a chain* $\mathbf{\Omega}$ *such that* $\mathbf{A}$ *can be embedded as an* ℓ*-monoid into* $\mathcal{R}es(\Omega)$*.*

We remark that if $\mathbf{\Omega}$ is a complete chain, in fact any complete lattice, then $\mathcal{R}es(\mathbf{\Omega})$ is a complete residuated lattice. However, the ℓ-monoid embeddings of the preceding two results do not always preserve the division operations.

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Francesco Paoli
Department of Pedagogy, Psychology, Philosophy
University of Cagliari
Via Is Mirrionis 1, 09123
Cagliari, Italy
paoli@unica.it

Constantine Tsinakis
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville, TN 37240, USA
constantine.tsinakis@vanderbilt.edu